Wave-Amplitude Transmission Matrix Approach to the Analysis of the Electromagnetic Planewave Reflection and Transmission at Multilayer Material Boundaries

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Abstract

This paper presents an approach to the problem of electromagnetic planewave reflection and transmission at multilayer material boundaries based on the wave-amplitude matrix method. In this approach, an analogy of the problem to an equivalent transmission line is evoked and the problem of electromagnetic reflection and transmission at material boundaries is modeled with a cascaded connection of a transmission line impedance step and a transmission line section. The generalized scattering matrix formulation is presented for a two-port network followed by the wave-amplitude matrix method for a cascaded two-port network. The wave-amplitude matrix method is then applied to the cascaded connection of transmission lines. Finally the proposed method is applied to the analysis of an anti-reflection coating and a half-wave radome.

Key words: Planewave, Material boundaries, Reflection & transmission, Wave-amplitude transmission matrix

I. Introduction

The problem of the planewave reflection and transmission at material boundaries has many practical applications such as the design of radomes [1]–[2] and optical anti-reflective or reflective coatings [3]–[4], multilayer metamaterial or photonic-crystal structures [5]–[6], and free-space dielectric measurements [7]. Undergraduate textbooks deal with the problem using electric and magnetic fields and provide formulas for the planewave reflection and transmission coefficients of a material boundary [8]–[9]. Optical multilayer coatings have traditionally been analyzed with the planewave transfer matrix, which relates the incident and reflected electric fields on both sides of the material boundary and the propagation of electric fields through a material layer. [3]–[4]. In the planewave transfer matrix formalism, matrix equations need to be separately derived for perpendicular and parallel polarizations.

In this paper, we present an analysis method for the planewave reflection and transmission at a multilayer structure, which is based on an equivalent transmission line and the wave-amplitude transmission matrix. Compared to the planewave transfer matrix, the proposed formulation handles both perpendicular and parallel polarizations with simple and unified equations.

First, a material boundary is modeled with an equivalent transmission line and the problem of the multilayer material structure is formulated in a cascaded transmission line, which is then analyzed using the wave-amplitude transmission matrix. Cascaded transmission lines can be analyzed either with the ABCD matrix or with the wave-amplitude matrix, which is also called the T-matrix. The wave-amplitude matrix is formulated in terms of incident and reflected waves in each layer so that it is a more intuitive tool than the ABCD matrix for analyzing planewave reflection and transmission at multilayer material boundaries.

Finally, as an example of the proposed method, the
reflection and transmission coefficients are obtained for a planewave incident on a material slab, on a quarter-wave anti-reflection coating and on a half-wave radome.

II. Formulation

Fig. 1 shows a boundary between two material half spaces with intrinsic impedances \( \eta_1 \) and \( \eta_2 \) where a planewave is incident from the half space 1 with an angle \( \theta_1 \).

![Fig. 1 Plane wave incident on a material boundary. TE or perpendicular polarization (left) and TM or parallel polarization (right)](image)

With the electric and magnetic fields of a planewave incident on the first layer, the fields in the second layer are obtained by enforcing the boundary conditions at the material boundary which requires the tangential electric and magnetic fields to be continuous across the boundary.

Assuming that a surface normal vector in Fig. 1 is along the positive \( z \) axis, the ratio of tangential electric and magnetic fields propagating in the positive \( z \) direction is given by

\[
\nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} \rightarrow \frac{E_{\tan}^+}{H_{\tan}^+} = \frac{\eta_1}{k_z} = Z_{\text{TE}} \tag{1a}
\]

\[
\nabla \times \mathbf{H} = j\omega \varepsilon \mathbf{E} \rightarrow \frac{E_{\tan}^+}{H_{\tan}^+} = \frac{k_z}{\omega \varepsilon} = Z_{\text{TM}} \tag{1b}
\]

for TE and TM polarizations. The superscript + is used to denote waves propagating in the positive \( z \) direction. For waves propagating in the negative \( z \) direction, the following relation holds.

\[
\frac{E_{\tan}^-}{H_{\tan}^-} = -Z_{\text{TE}} \text{ OR } -Z_{\text{TM}} \tag{2}
\]

\( k_z \) is the wavenumber in the \( z \) direction given by

\[
k_z = \sqrt{k^2 - (k_1 \sin \theta_1)^2} = \beta - j\alpha \tag{3}
\]

where \( k_1 \) and \( \theta_1 \) are the wavenumber and incidence angle of the first material. The material constants are cast in a general form to include lossy materials.

\[
\mu = \mu' - j\mu''; \quad \varepsilon = \varepsilon' - j\varepsilon'' = \frac{\sigma}{\omega} \tag{4}
\]

Continuity of the tangential electric and magnetic fields at the boundary yields the following equations for the TE polarization:

\[
E_{\tan,1}^+ + E_{\tan,1}^- = E_{\tan,2}^+ \tag{5}
\]

\[
\frac{E_{\tan,1}^+}{Z_1} - \frac{E_{\tan,1}^-}{Z_1} = \frac{E_{\tan,2}^+}{Z_2} \tag{6}
\]

where \( Z_1 \) and \( Z_2 \) are the wave impedance given by (1a).

\[
Z_i = Z_{\text{TE,i}} \quad (i = 1,2) \tag{7}
\]

Similarly for the TM polarization, the following equations hold.

\[
H_{\tan,1}^+ + H_{\tan,1}^- = H_{\tan,2}^+ \tag{8}
\]

\[
Z_i H_{\tan,1}^+ - Z_i H_{\tan,1}^- = Z_2 H_{\tan,2}^+ \tag{9}
\]

where \( Z_1 \) and \( Z_2 \) are the wave impedance given by (1b).

\[
Z_i = Z_{\text{TM,i}} \quad (i = 1,2) \tag{10}
\]

An equivalent transmission line model is obtained by associating the tangential electric and magnetic fields with the voltage and current waves respectively on a transmission line:

\[
E_{\tan}^+ \leftrightarrow V^+; \quad H_{\tan}^+ \leftrightarrow I^+ \tag{11}
\]

Thus for the TE polarization, we obtain

\[
V_1^+ + V_1^- = V_2^+ \tag{12}
\]

\[
\frac{V_1^+}{Z_1} - \frac{V_1^-}{Z_1} = \frac{V_2^+}{Z_2} \tag{13}
\]

while for the TM polarization, the corresponding equations are given by

\[
I_1^+ + I_1^- = I_2^+ \tag{14}
\]

\[
Z_1 I_1^+ - Z_1 I_1^- = Z_2 I_2^+ \tag{15}
\]

Eqs. (14) and (15) are reduced to (13) and (12) respectively with the following substitution.

\[
I_1^+ = \frac{V_1^+}{Z_1}; \quad I_1^- = \frac{V_1^-}{Z_1}; \quad I_2^+ = \frac{V_2^+}{Z_2} \tag{16}
\]

Eqs. (12) to (15) are same as those for an impedance step discontinuity on a transmission line shown in Fig. 2. The problems of Fig. 1 and Fig. 2 are equivalent since the tangential electric and magnetic fields in the first problem and the voltage and current waves in the second problem obey the same Maxwell's equations and the same continuity conditions at the material interface.

![Fig. 2 Transmission line model for the planewave reflection and transmission at a material boundary](image)
The problem of the planewave reflection and transmission at multilayer material boundaries can be cast in a cascaded connection of transmission lines where each material layer is replaced with a transmission line section and the tangential electric and magnetic fields are replaced with the voltage and current waves on the transmission line. To complete an equivalent transmission line model, we replace each material layer with a transmission line section whose length is same as the thickness of the layer and whose characteristic impedance is same as the wave impedance given by (1a) and (1b), and whose propagation constant is same as the wavenumber in the z direction given by (3).

Once the tangential electric field in each layer is obtained, the tangential magnetic field and the total electric and magnetic fields can be found using

\[ H^+_{\text{tan}} = E^+_{\text{tan}}/Z_{\text{TE}}; \quad E^+ = E^+_{\text{tan}}; \quad H^+ = E^+/\eta \quad \text{(TE)} \]

\[ H^+_{\text{tan}} = E^+_{\text{tan}}/Z_{\text{TM}}; \quad H^+ = H^+_{\text{tan}}; \quad E^+ = \eta H^+ \quad \text{(TM)} \]

With the equivalence between the multilayer material structure and the cascaded transmission line established, the next step is to analyze the cascaded transmission line using the wave-amplitude transmission matrix. We start with the generalized scattering matrix of a two-port network shown in Fig. 3.

![Fig. 3](image_url)

**Fig. 3** Two-port network parameters with voltage waves (left) and corresponding normalized voltage waves (right)

The normalized voltage waves are defined by

\[ a_i = \frac{V_i^+}{\sqrt{Z_i}}; \quad b_i = \frac{V_i^-}{\sqrt{Z_i}} \quad \text{(19a)} \]

\[ a_2 = \frac{V_2^+}{\sqrt{Z_2}}; \quad b_2 = \frac{V_2^-}{\sqrt{Z_2}} \quad \text{(19b)} \]

where \( Z_i \) is the characteristic impedance of a transmission line connected to the port \( i \), the superscripts \( i \) and \( r \) denote the voltage wave incident and reflected at the port while \( + \) and \( - \) the waves traveling in the right and left directions. The scattering parameters are defined by the following equations

\[ b_1 = S_1 a_1 + S_2 a_2 \quad \text{(20a)} \]

\[ b_2 = S_2 a_1 + S_1 a_2 \quad \text{(20b)} \]

The equation (20) can be written in a matrix form:

\[ [b] = [S][a] \quad \text{(21)} \]

The wave-amplitude matrix description of a two-port network is shown in Fig. 4. The wave-amplitude matrix is a rearrangement of the scattering matrix for the analysis of a cascaded two-port network:

\[ c_1^+ = a_1 = \frac{V_1^+}{\sqrt{Z_1}}; \quad c_1^- = b_1 = \frac{V_1^-}{\sqrt{Z_1}} \quad \text{(22a)} \]

\[ c_2^+ = b_2 = \frac{V_2^+}{\sqrt{Z_2}}; \quad c_2^- = a_2 = \frac{V_2^-}{\sqrt{Z_2}} \quad \text{(22b)} \]

![Fig. 4](image_url)

**Fig. 4** Two-port network described by wave amplitudes

\[ \begin{bmatrix} c_1^+ \\ c_2^- \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} \]

The equation (23) can be written in a matrix form:

\[ [c_1] = [A][c_2] \quad \text{(24)} \]

The elements of the matrix \([A]\) can be expressed in terms of the scattering parameters.

\[ A_{11} = \frac{b_1}{a_2 b_{12} - a_1 b_{21}} = \frac{S_{12}}{S_{21}} \quad \text{(25a)} \]

\[ A_{21} = \frac{b_2}{a_2 b_{12} - a_1 b_{21}} = \frac{S_{11}}{S_{21}} \quad \text{(25b)} \]

Eq. (25) can be written in a matrix form.

\[ [A] = \begin{bmatrix} 1 & -S_{22} \\ S_{21} & S_{12} S_{21} - S_{11} S_{22} \end{bmatrix} \quad \text{(26)} \]

The cascaded two-port network shown in Fig. 5 is now easily analyzed by multiplying the transmission matrix of each two-port network.

\[ [c_1] = [A_1][c_2] \ldots [A_N] \quad \text{(27)} \]

![Fig. 5](image_url)

**Fig. 5** Wave-amplitude transmission matrix representation of a cascaded two-port network

Fig. 6 shows a multilayer material structure and its equivalent transmission line representation. In Fig. 6, and \( s_i \) are the impedance given by (1a) or (1b) and the wavenumber given by (3) respectively. To analyze the cascaded transmission line shown in Fig. 6, expressions are derived for the wave-amplitude transmission matrix for
an impedance step and for a transmission line section.

Fig. 7 shows a step in the characteristic impedance of a transmission line. The continuity of the total voltage and the total current implies

\[
\eta_1 \eta_2 \ldots \eta_{N-1} \eta_N
\]
\[
d_2 \ldots d_{N-1}
\]
\[
Z_1 Z_2 \ldots Z_{N-1} Z_N
\]
\[
k_{22} k_{23} \ldots k_{N-2} k_{N-1}
\]
\[
d_2 \ldots d_{N-1}
\]

Fig. 6 Multilayer material structure and the equivalent cascaded transmission line

\[V_1 = V_2 \rightarrow V_1' + V_2' = V_2' + V_2'' \quad (29)\]
\[I_1 = I_2 \rightarrow V_1' \frac{Z_2}{Z_1} - V_2' \frac{Z_2}{Z_1} = V_2'' \frac{Z_1}{Z_2} - V_2'' \frac{Z_1}{Z_2} \quad (30)\]

from which scattering parameters are obtained as follows.

\[S_{11} = \frac{V_1'}{V_1'} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = -S_{22} = \frac{V_2''}{V_2''} \quad (31a)\]
\[S_{21} = \frac{V_2'}{V_1'} = \frac{2Z_1Z_2}{Z_2 + Z_1} = S_{12} = \frac{V_1'}{V_2'} \quad (31b)\]

It is convenient to treat a combination of an impedance step and a transmission line section shown in Fig. 8 as a single unit. The transmission matrix of the combined structure is given by

\[
[A] = \frac{1}{T} \begin{bmatrix} 1 & R_1 e^{j\phi} \ e^{j\phi} & 0 \\ R_1 & 0 & e^{-j\phi} \end{bmatrix} = \frac{1}{T} \begin{bmatrix} e^{j\phi} & Re^{j\phi} \\ e^{-j\phi} & e^{j\phi} \end{bmatrix} \quad (34)\]

where

\[
R = \frac{Z_2 - Z_1}{Z_2 + Z_1}; T = \frac{2\sqrt{Z_2Z_1}}{Z_2 + Z_1} \quad (35)\]

As an example of the application of the proposed method, we find the transmission and reflection coefficients of the material coating shown in Fig. 9, where the region 2 is a coating layer.

\[
[A] = \frac{1}{T_1} \begin{bmatrix} 1 & R_1 \ e^{j\phi} & 0 \\ R_1 & 0 & e^{-j\phi} \end{bmatrix} \frac{1}{T_2} \begin{bmatrix} 1 & R_2 \ e^{j\phi} & 0 \\ R_2 & 0 & e^{-j\phi} \end{bmatrix} \quad (36)\]
\[
[A] = \frac{1}{T_1T_2} \begin{bmatrix} 1 & R_1e^{j\phi} & e^{j\phi} \\ R_1e^{-j\phi} & e^{-j\phi} & e^{j\phi} \end{bmatrix} \quad (37)\]

where

\[
R_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}; T_1 = \frac{2\sqrt{Z_2Z_1}}{Z_2 + Z_1} \quad (38)\]
\[
R_2 = \frac{Z_3 - Z_2}{Z_3 + Z_2}; T_2 = \frac{2\sqrt{Z_3Z_2}}{Z_3 + Z_2} \quad (39)\]

\[
\phi = k_{c2}d \quad (40)\]

Fig. 7 Step in the characteristic impedance of a transmission line (left) and a transmission line section (b)

The transmission matrix of a transmission line section is given by

\[
[A] = \begin{bmatrix} e^{j\phi} & 0 \\ 0 & e^{-j\phi} \end{bmatrix} \quad (32)\]

where the electrical length \( \phi \) is given by

\[
\phi = k_{c2}d \quad (33)\]

which is in general a complex number.

\[
\frac{c_3}{c_1} \phi \quad (41)\]

Fig. 8 Unit cell in a cascaded transmission line.

For an infinite medium 3, there will be no reflected wave in the region 3 so that we have

\[
c_3 = 0 \quad (42)\]

Thus we obtain

\[
\frac{c_3}{c_1} = \frac{E_{in,3}}{E_{in,1}} / \sqrt{\frac{Z_2}{Z_1}} = \frac{T_1T_2}{R_1R_2e^{j\phi} + e^{j\phi}} \quad (42)\]
Then the plane wave reflection and transmission coefficients are given by

\[
R_p = \frac{E_{\text{tan},1}}{E_{\text{tan},1}^*} = \frac{R_3 e^{-j\theta} + R_2 e^{j\theta}}{R_1 R_3 e^{-j\theta} + e^{j\theta}}
\]

(43)

and

\[
T_p = \frac{E_{\text{tan},3}^*}{E_{\text{tan},1}^*} = \sqrt{\frac{R_3}{Z_1}} \frac{T_1 T_2}{Z_1} = \frac{T_2}{Z_1}
\]

(44)

For a quarter-wave anti-reflection coating or a quarter-wave transformer, we require

\[R_1 = R_2 = R
\]

(46)

Then the reflection coefficient is reduced to

\[
R_p = \frac{2R\cos \phi}{R^2 e^{-j\theta} + e^{j\theta}}
\]

(47)

From (46) we obtain

\[Z_2 = \sqrt{Z_1 Z_3}
\]

(48)

while from (47) we obtain

\[
\phi = n\pi + \pi / 2 \quad (n = 0, 1, \ldots)
\]

(49)

The reflection and transmission coefficients of a quarter-wave transformer are now given by the following equations.

\[
R_q = \frac{E_{\text{tan},1}}{E_{\text{tan},1}^*} = \frac{(Z_3 - Z_1) \cos \phi}{(Z_3 + Z_1) \cos \phi + j2\sqrt{Z_1 Z_3} \sin \phi} = 0
\]

(50)

\[
T_q = \frac{E_{\text{tan},3}^*}{E_{\text{tan},1}^*} = \sqrt{\frac{Z_3}{Z_1}} \frac{2\sqrt{Z_1 Z_3} \cos \phi + j2\sqrt{Z_1 Z_3} \sin \phi}{(Z_3 + Z_1) \cos \phi + j2\sqrt{Z_1 Z_3} \sin \phi}
\]

\[
= \frac{Z_1}{Z_3} \frac{1}{j \sin \phi} = \pm j \frac{Z_3}{Z_1}
\]

(51)

For a material slab where the material is same for the regions 1 and 2, the following holds

\[Z_3 = Z_1
\]

(52)

\[R_1 = -R_2 = R = \frac{Z_3 - Z_1}{Z_3 + Z_1}
\]

(53)

\[T_1 = T_2 = T = \frac{2\sqrt{Z_1 Z_3}}{Z_3 + Z_1}
\]

(54)

\[
[A] = \frac{1}{T}\begin{bmatrix}
-R_3 e^{-j\theta} + e^{j\theta} & R_3 e^{j\theta} - Re^{j\theta} \\
-R_2 e^{-j\theta} + R_2 e^{j\theta} & e^{-j\theta} - R_2 e^{j\theta}
\end{bmatrix}
\]

(55)

\[
\frac{c_1}{c_3} = \frac{E_{\text{tan},3}^*}{E_{\text{tan},1}^*} = \frac{T^2}{-R_3 e^{-j\theta} + e^{j\theta}}
\]

(56)

\[
\frac{c_1}{c_3} = \frac{E_{\text{tan},3}^*}{E_{\text{tan},1}^*} = \frac{T^2}{-R_2 e^{-j\theta} + R_2 e^{j\theta}}
\]

(57)

\[R_S = \frac{E_{\text{tan},1}}{E_{\text{tan},1}^*} = \frac{-Re^{-j\theta} + Re^{j\theta}}{-R^2 e^{-j\theta} + e^{j\theta}}
\]

(58)

\[
T_S = \frac{E_{\text{tan},3}^*}{E_{\text{tan},1}^*} = \frac{-R^2 e^{-j\theta} + e^{j\theta}}{T^2}
\]

(59)

For a matched single-wall radome, we have

\[
\phi = n\pi \quad (n = 1, 2, \ldots)
\]

(60)

\[
R_S = \frac{E_{\text{tan},1}}{E_{\text{tan},1}^*} = \frac{j(Z_2^2 - Z_1^2) \sin \phi}{j(Z_2^2 + Z_1^2) \sin \phi + 2Z_1 \cos \phi} = 0
\]

(61)

\[
T_S = \frac{E_{\text{tan},3}^*}{E_{\text{tan},1}^*} = \frac{2Z_1 Z_3}{j(Z_2^2 + Z_1^2) \sin \phi + 2Z_1 \cos \phi} = \pm j
\]

(62)

### III. Conclusion

The electric and magnetic fields tangential to a material boundary can be associated with the voltage and current waves respectively at a transmission line step discontinuity since they obey the same boundary conditions and the same governing equations. A multilayer material structure then can be modeled as a cascaded connection of impedance steps and transmission line sections, which has been analyzed using the wave-amplitude transmission matrix. The proposed method is simple and can handle lossy materials without any modification. Expressions for the plane wave reflection and transmission coefficients for a material coating problem have been derived and applied to the quarter-wave anti-reflection coating and to the half-wave radome. The method presented in this paper may find applications in the design and analysis of multilayer electromagnetic structures.

### References


