LECTURE 20: MICROSTRIP ANTENNAS – PART I
(Introduction. Construction and geometry. Feeding techniques. Substrate properties. Loss calculation.)

1. Introduction

Microstrip antennas (MSA) received considerable attention in the 1970’s, although the first designs and theoretical models appeared in the 1950’s. They are suitable for many mobile applications: handheld devices, aircraft, satellite, missile, etc. The MSA are low profile, mechanically robust, inexpensive to manufacture, compatible with MMIC designs and relatively light and compact. They are quite versatile in terms of resonant frequencies, polarization, pattern and impedance. They allow for additional tuning elements like pins or varactor diodes between the patch and the ground plane.

Some of the limitations and disadvantages of the MSA are:

- relatively low efficiency (due to dielectric and conductor losses)
- low power
- spurious feed radiation (surface waves, strips, etc.)
- narrow frequency bandwidth (at most a couple of percent)
- relatively high level of cross polarization radiation

MSA are applicable in the GHz range \( f > 0.5 \text{ GHz} \). For lower frequencies their dimensions become too large.
2. Construction and Geometry

Generally the MSA are thin metallic patches of various shapes etched on dielectric substrates of thickness \( h \), which usually is from \( 0.003 \lambda_0 \) to \( 0.05 \lambda_0 \). The substrate is usually grounded at the opposite side.

The dimensions of the patch are usually in the range from \( \lambda_0/3 \) to \( \lambda_0/2 \). The dielectric constant of the substrate \( \varepsilon_r \) is usually in the range from 2.2 to 12. The most common designs use relatively thick substrates with lower \( \varepsilon_r \) because they provide better efficiency and larger bandwidth. On the other hand, this implies larger dimensions of the antennas. The choice of the substrate is limited by the RF or microwave circuit coupled to the antenna, which has to be built on the same board. The microwave circuit together with the antenna is usually manufactured by photo-etching technology.
Types of microstrip radiators

(a) Single radiating patches

(b) Single slot radiator

The feeding microstrip line is beneath (etched on the other side of the substrate) – see dash-line.
(c) Microstrip traveling wave antennas

The open end of the long TEM line is terminated in a matched resistive load.
(d) Microstrip antenna arrays
3. Feeding Methods

1) Microstrip feed – easy to fabricate, simple to match by controlling the inset position and relatively simple to model. However, as the substrate thickness increases, surface waves and spurious feed radiation increase.

2) Coaxial probe feed – easy to fabricate, low spurious radiation; difficult to model accurately; narrow bandwidth of impedance matching.
3) Aperture coupling (no contact), microstrip feed line and radiating patch are on both sides of the ground plane, the coupling aperture is in the ground plane – low spurious radiation, easy to model; difficult to match, narrow bandwidth.

4) Proximity coupling (no contact), microstrip feed line and radiating patch are on the same side of the ground plane – largest bandwidth (up to 13%), relatively simple to model, has low spurious radiation.
More examples of microstrip and coaxial probe feeds:

**STRIP FEEDS**

**COAX FEEDS**
4. Surface Waves

Surface waves can be excited at the dielectric-to-air interface. Surface waves give rise to end-fire radiation. In addition they can lead to unwanted coupling between array elements. The phase velocity of the surface waves is strongly dependent on the dielectric constant $\varepsilon_r$ and thickness $h$ of the substrate. The excitation of surface waves in a dielectric slab backed by a ground plane has been well studied (Collin, *Field Theory of Guided Waves*). The lowest-order TM mode, TM$_0$, has no cut-off frequency. The cut-off frequencies for the higher-order modes (TM$_n$ and TE$_n$) are given by

$$f_{c}^{(n)} = \frac{n \cdot c}{4h\sqrt{\varepsilon_r - 1}}, \quad n = 1, 2, \ldots,$$

(1)

where $c$ is the speed of light. The cut-off frequencies for the TE$_n$ modes are given by the odd $n = 1, 3, 5, \ldots$, and the cut-off frequencies for the TM$_n$ modes are given by the even $n$. For the TE$_1$ mode the calculated values of $h / \lambda_c^{(1)}$ are $[\lambda_c^{(1)} = c / f_c^{(1)}$, $h / \lambda_c^{(1)} = n / (4\sqrt{\varepsilon_r - 1})]$

a) 0.217 for duroid ($\varepsilon_r = 2.32$),
b) 0.0833 for alumina ($\varepsilon_r = 10$).

Thus, the lowest-order TE$_1$ mode is excited at 41 GHz for 1.6 mm thick duroid substrate, and at about 39 GHz for 0.635 mm thick alumina substrate. The substrate thickness is chosen so that the ratio $h / \lambda_0$ is well below $h / \lambda_c^{(1)}$ ($\lambda_0$ is the free-space wavelength at the operating frequency), i.e., [3]

$$h < \frac{c}{4f_u\sqrt{\varepsilon_r - 1}},$$

(2)

where $f_u$ is the highest frequency in the band of operation. Note that $h$ should be chosen as high as possible under the constraint of (2), so that maximum efficiency is achieved. Also, $h$ has to conform to the commercially available substrates. Another practical formula for $h$ is given in [2]:

$$h \leq \frac{0.3c}{2\pi f_u\sqrt{\varepsilon_r}}.$$  

(3)

The TM$_0$ mode has no cut-off frequency and is always present to some extent. The surface TM$_0$ wave excitation becomes appreciable when $h / \lambda > 0.09$ ($\varepsilon_r \approx 2.3$) and when $h / \lambda > 0.03$ ($\varepsilon_r \approx 10$). Generally, to suppress the TM$_0$
mode, the dielectric constant should be lower and the substrate height should be smaller. Unfortunately, decreasing $\varepsilon_r$ increases the antenna size, while decreasing $h$ leads to smaller antenna efficiency and frequency band.

5. Criteria for Substrate Selection

1) surface-wave excitation
2) dispersion of the dielectric constant and loss tangent of the substrate
3) copper loss
4) anisotropy in the substrate
5) effects of temperature, humidity, and aging
6) mechanical requirements: conformability, machinability, solderability, weight, elasticity, etc.
7) cost

The first 3 factors are of special concern in the millimeter-wave range ($f \geq 30$ GHz).

<table>
<thead>
<tr>
<th>Material</th>
<th>Dielectric Constant</th>
<th>Loss Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unreinforced PTFE, Cuflon</td>
<td>2.1</td>
<td>0.0004</td>
</tr>
<tr>
<td>Reinforced PTFE, RT Duroid 5880</td>
<td>2.20 (1.5%)</td>
<td>0.0009</td>
</tr>
<tr>
<td>Fused Quartz</td>
<td>3.78</td>
<td>0.0001</td>
</tr>
<tr>
<td>96% Alumina</td>
<td>9.40 (5%)</td>
<td>0.0010</td>
</tr>
<tr>
<td>99.5% Alumina</td>
<td>9.80 (5%)</td>
<td>0.0001</td>
</tr>
<tr>
<td>Sapphire</td>
<td>9.4, 1.6</td>
<td>0.0001</td>
</tr>
<tr>
<td>Semi-Insulating GaAs</td>
<td>12.9</td>
<td>0.0020</td>
</tr>
</tbody>
</table>
### NON-ELECTRICAL PROPERTIES OF COMMONLY USED SUBSTRATE MATERIALS FOR MICROSTRIP ANTENNAS

<table>
<thead>
<tr>
<th>Properties</th>
<th>PTFE</th>
<th>Fused Quartz</th>
<th>Alumina</th>
<th>Sapphire</th>
<th>GaAs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>temperature range (°C)</strong></td>
<td>-55 – 260</td>
<td>&lt; +1100</td>
<td>&lt; +1600</td>
<td>-24 – 370</td>
<td>-55 – 260</td>
</tr>
<tr>
<td><strong>Thermal conductivity (W/cm·K)</strong></td>
<td>0.0026</td>
<td>0.017</td>
<td>0.35 to 0.37</td>
<td>0.42</td>
<td>0.46</td>
</tr>
<tr>
<td><strong>Coefficient of thermal expansion (ppm/K)</strong></td>
<td>16.0 to 108.0</td>
<td>0.55</td>
<td>6.30 to 6.40</td>
<td>6.00</td>
<td>5.70</td>
</tr>
<tr>
<td><strong>Temperature coefficient of dielectric constant (ppm/K)</strong></td>
<td>+350.0 to +480.0</td>
<td>+13.0</td>
<td>+136.0</td>
<td>+110 to +140</td>
<td>-</td>
</tr>
<tr>
<td><strong>Minimum thickness (mil)</strong></td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td><strong>Machinability</strong></td>
<td>Good</td>
<td>Very poor</td>
<td>Very poor</td>
<td>Poor</td>
<td>Poor</td>
</tr>
<tr>
<td><strong>Solderability</strong></td>
<td>Good</td>
<td>Good</td>
<td>Good</td>
<td>Good</td>
<td>Good</td>
</tr>
<tr>
<td><strong>Dimensional Stability</strong></td>
<td>Poor for unreinforced, very good for others</td>
<td>Good</td>
<td>Excellent</td>
<td>Good</td>
<td>Good</td>
</tr>
<tr>
<td><strong>Cost</strong></td>
<td>Very low</td>
<td>High</td>
<td>Low</td>
<td>-</td>
<td>Very high</td>
</tr>
</tbody>
</table>
6. Dispersion Effects in the Substrate

The dependence of the dielectric constant $\varepsilon_r$ and the loss tangent on frequency is referred to as frequency dispersion. For frequencies up to 100 GHz (the typical range for printed antennas is $< 30$ GHz), the dispersion of $\varepsilon_r$ is practically negligible. The losses, however, display noticeable changes with frequency. In general, the loss increases with frequency.

7. Dielectric loss and copper loss

The loss in the feed lines and the patches themselves are usually computed with formulas, which were first derived for microstrip transmission lines.

a) Dielectric loss (in dB per unit length, length is in the units used for $\lambda_0$)

$$\alpha_d = 27.3 \cdot \frac{\varepsilon_r}{\sqrt{\varepsilon_{\text{eff}}(f)}} \cdot \frac{\varepsilon_{\text{eff}}(f) - 1}{(\varepsilon_r - 1)} \cdot \tan \delta \cdot \frac{\varepsilon_{\text{eff}}(f)}{\lambda_0}$$  \hspace{1cm} (4)

b) Copper loss (in dB per unit length)

$$\alpha_c = \left\{ \begin{array}{ll}
1.38 \cdot \frac{R'_s}{h Z_0} \cdot \frac{32 - \left( \frac{W'}{h} \right)^2}{32 + \left( \frac{W'}{h} \right)^2} \Lambda, & \text{for } \frac{W}{h} \leq 1 \\
6.1 \times 10^{-5} \cdot \frac{R'_s Z_0 \varepsilon_{\text{eff}}(f)}{h} \cdot \left[ \left( \frac{W'}{h} \right) + \frac{0.667 W'}{h} \right] \Lambda, & \text{for } \frac{W}{h} \geq 1
\end{array} \right.$$

\hspace{1cm} (5)
In the above equations:

- $\varepsilon_{r_{\text{eff}}}(f)$ is the effective dielectric constant (generally, dispersive). Its quasi-static (low frequency) expression [2] is

\[
\varepsilon_{r_{\text{eff}}}(0) = \begin{cases} 
\frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2} \cdot \left(1 + 12 \frac{h}{W}\right)^{-1/2}, & \text{for } W / h > 1 \\
\frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2} \cdot \left(1 + 12 \frac{h}{W}\right)^{-1/2} + 0.04 \left(1 - \frac{W}{h}\right)^2, & \text{for } W / h \leq 1 
\end{cases}
\]

(6)

Alternative expression for the quasi-static approximation of $\varepsilon_{r_{\text{eff}}}$ can be found in [5].

The quasi-static expressions need a dispersion correction for frequencies higher than 8 GHz. One possible correction is based on an empirical formula for the dispersive phase velocity in a microstrip line [5]. We first compute a normalized frequency (normalized with respect to the cut-off of the TE\textsubscript{1} mode):

\[
f_n = \frac{f}{f_{c(1)}} = \frac{4h \sqrt{\varepsilon_r - 1}}{\lambda_0}.
\]

(7)

Then, the dispersive phase velocity is calculated as

\[
\nu_p = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_{r_{\text{eff}}}(0)}} \cdot \frac{f_n^2 \sqrt{\varepsilon_{r_{\text{eff}}}(0)} + \sqrt{\varepsilon_r}}{f_n^2 + 1}.
\]

(8)

Finally, 

\[
\varepsilon_{r_{\text{eff}}}(f) = \left(\frac{c}{\nu_p}\right)^2.
\]

(9)

For alternative formulas, refer to [5].

- $Z_0$ is the characteristic impedance of the microstrip line (generally, dispersive):
\[ Z_0 = \begin{cases} \frac{120\pi \sqrt{\varepsilon_{\text{eff}}}}{W/h + 1.393 + 0.667 \ln \left( \frac{W}{h} + 1.444 \right)}, & \text{for } \frac{W}{h} \geq 1 \\ \frac{60}{\sqrt{\varepsilon_{\text{eff}}}} \cdot \ln \left( \frac{8h}{W} + 0.25 \frac{W}{h} \right), & \text{for } \frac{W}{h} \leq 1 \end{cases} \]  

(10)

- \( \Lambda \) is a constant dependent on the strip thickness \( t \):

\[ \Lambda = \begin{cases} 1 + \frac{h}{W'} \left[ 1 + \frac{1.25t}{\pi W} + \frac{1.25}{\pi} \ln \left( \frac{4\pi W}{t} \right) \right], & \text{for } \frac{W}{h} \leq \frac{1}{2\pi} \\ 1 + \frac{h}{W'} \left[ 1 - \frac{1.25t}{\pi W} + \frac{1.25}{\pi} \ln \left( \frac{2t}{t} \right) \right], & \text{for } \frac{W}{h} \geq \frac{1}{2\pi} \end{cases} \]  

(11)

- \( W' \) is the effective strip width:

\[ \frac{W'}{h} = \begin{cases} \frac{W}{h} + \frac{1.25t}{\pi h} \left[ 1 + \ln \left( \frac{4\pi W}{t} \right) \right], & \text{for } \frac{W'}{h} \leq \frac{1}{2\pi} \\ \frac{W}{h} + \frac{1.25t}{\pi h} \left[ 1 + \ln \left( \frac{2h}{t} \right) \right], & \text{for } \frac{W'}{h} \geq \frac{1}{2\pi} \end{cases} \]  

(12)

- \( R'_s \) is the effective surface resistance of the conductor:

\[ R'_s = R_s \left\{ 1 + \frac{2}{\pi} \arctan \left[ 1.4 \left( \frac{\Lambda}{\delta} \right)^2 \right] \right\}, \Omega \]  

(13)

where \( R_s = \sqrt{\pi f \mu / \sigma} \) is the high-frequency surface resistance of the conductor. \( R_s \) relates to the skin-depth \( \delta \) as \( R_s = (\delta \sigma)^{-1} \). For a uniform surface current distribution over a conducting rod of length \( l \) and perimeter of its cross-section \( P \), the resultant resistance is

\[ R_{hf} = R_s \cdot l / P, \Omega. \]  

(14)

Finally, the total loss is the sum of the conduction and dielectric losses:

\[ \alpha_t = \alpha_d + \alpha_c. \]  

(15)
SURFACE RESISTANCE AND SKIN-DEPTH OF COMMONLY USED CONDUCTORS

<table>
<thead>
<tr>
<th>Metal</th>
<th>Rs [Ohm/square x10⁻⁷]</th>
<th>Skin-depth at 2 GHz [µm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>2.5</td>
<td>1.4</td>
</tr>
<tr>
<td>Copper</td>
<td>2.6</td>
<td>1.5</td>
</tr>
<tr>
<td>Gold</td>
<td>3.0</td>
<td>1.7</td>
</tr>
<tr>
<td>Aluminum</td>
<td>3.3</td>
<td>1.9</td>
</tr>
</tbody>
</table>

Some References

1. Transmission line model – the rectangular patch

The TL model is the simplest of all, representing the rectangular patch as a parallel-plate transmission line connecting two radiating slots (apertures), each of width $W$ and height $h$. In the figure below, $z$ is the direction of propagation of the transmission line.

The TL model is not accurate and lacks versatility. However, it gives a relatively good physical insight into the nature of the patch antenna and the field distribution for all $\text{TM}_{00n}$ modes.

The slots represent very high-impedance terminations from both sides of the transmission line (almost an open circuit). Thus, we expect this structure to have highly resonant characteristics depending crucially on its length $L$ along $z$. The resonant length of the patch, however, is not exactly equal to the physical length due to the fringing effect. The fringing effect makes the effective electrical length of the patch longer than its physical length,
$L_{\text{eff}} > L$. Thus, the resonance condition $\beta^{(n)} L_{\text{eff}} = n \cdot \pi / 2$, $n = 1, 2, \ldots$, depends on $L_{\text{eff}}$, not $L$. A sketch of the E-field distribution for the first (dominant) resonant mode, $n = 1$, is shown in the figure below.

A. Computing the effective patch length

$$\frac{\Delta L}{h} = 0.412 \frac{(\varepsilon_{\text{eff}} + 0.3) \left( \frac{W}{h} + 0.264 \right)}{ \left( \varepsilon_{\text{eff}} - 0.258 \right) \left( \frac{W}{h} + 0.8 \right)}.$$  \hspace{1cm} (21.1)

For the computation of $\varepsilon_{\text{eff}}$, see previous Lecture. The effective length is

$$L_{\text{eff}} = L + 2\Delta L.$$  \hspace{1cm} (21.2)
B. Resonant frequency of the dominant TM\(_{001}\) mode

\[
L_{eff} = \frac{\lambda_0}{2} = \frac{v}{2f^{(001)}} = \frac{c}{2\sqrt{\epsilon_{eff}} f^{(001)}} \Rightarrow f_r^{(001)} = \frac{c}{2L_{eff} \sqrt{\epsilon_{eff}}} \tag{21.3}
\]

The resonant frequency of a patch depends strongly on \(L\), therefore, the exact calculation of \(L_{eff}\) is necessary to predict the antenna resonance:

\[
f_r^{(001)} = \frac{c}{2 \sqrt{\epsilon_{eff}} \left( L + 2\Delta L \right)} \tag{21.4}
\]

The field of the TM\(_{001}\) mode does not depend on the \(x\) and \(y\) coordinates but it strongly depends on the \(z\) coordinate, along which a standing wave is formed. The figure below shows the vertical \(E\)-field distribution along \(z\) when the patch is in resonance.
C. The patch width $W$

\[ W = \frac{1}{2 f_r \sqrt{\mu_0 \varepsilon_0}} \sqrt{\frac{2}{\varepsilon_r + 1}} = \frac{c}{2 f_r \sqrt{\varepsilon_r + 1}} \]  

Expression (21.5) makes the width $W$ equal to about half a wavelength. It leads to good radiation efficiencies and acceptable dimensions. Thus, the patch can be viewed as a continuous planar source consisting of infinite number of infinitesimally thin half-wavelength dipoles.

D. Equivalent circuit of the patch

The dominant TM$_{001}$ mode has a uniform field distribution along the $y$-axis at the slots formed at the front and end edges of the patch. The equivalent conductance $G$ is obtained from the theory of uniform apertures while $B$ is related to the fringe capacitance:

\[ G = \frac{W}{120 \lambda_0} \left[ 1 - \frac{1}{24} \left( \frac{2\pi h}{\lambda_0} \right)^2 \right], \quad \text{for } \frac{h}{\lambda_0} < \frac{1}{10}, \]  

\[ B = \frac{W}{120 \lambda_0} \left[ 1 - 0.636 \ln \left( \frac{2\pi h}{\lambda_0} \right)^2 \right], \quad \text{for } \frac{h}{\lambda_0} < \frac{1}{10}. \]  

The limitation $h / \lambda_0 < 0.1$ is necessary since a uniform field distribution along the $x$-axis is assumed. The patch has two radiating slots (see the figure below).
The equivalent circuit of a slot is constructed as a parallel $R-C$ circuit, using the values computed by (21.6) and (21.7):

$$
\begin{align*}
B & \quad \parallel \quad G
\end{align*}
$$

$G = 1/R$ represents the radiation loss, while $B = j\omega C$ is the equivalent susceptance, which represents the capacitance of the slot.

More accurate values for the conductance $G$ can be obtained through the cavity model:

$$
G = \frac{I}{120\pi^2}, \quad (21.8)
$$

where

$$
I = \pi \left[ \sin \left( \frac{k_0 W \cos \theta}{2} \right) \right]^2 \sin^3 \theta d\theta = -2 + \cos X + X \cdot S_i(X) + \frac{\sin X}{X}, \quad (21.9)
$$

and $X = k_0 W$, $k_0 = \omega \sqrt{\mu_0 \varepsilon_0}$. $S_i$ denotes the sine integral, $S_i(x) = \int_0^x (\sin y) / y dy$.

The equivalent circuit representing the whole patch in the TM$_{001}$ mode includes the two radiating slots as parallel $R-C$ circuits and the patch connecting them as a transmission line whose characteristics are computed in the same way as those of a microstrip transmission line.
Here, $Z_c$ is the characteristic impedance of the line, and $\beta_g$ is its phase constant. When the losses are not neglected, we must include also the attenuation constant $\alpha$ (see Lecture 20). For each slot, $G$ represents the radiation loss and $B = \omega C$ represents the capacitance associated with the fringe effect.

E. Resonant input resistance

When the patch is resonant, the susceptances of both slots cancel out at the feed point regardless of the position of the feed along the patch. Thus, the input admittance is always purely real. This real value, however, strongly depends on the feed position along $z$. This is easily shown through the Smith chart for the admittance transformation through a transmission line.

At the feed point, the impedance of each slot is transformed by the respective transmission line representing a portion of the patch:

\[
Y_{in} = Y_1 + Y_2
\]  

(21.10)

The admittance transformation is given by

\[
Y_{in} = Y_c \frac{Y_L + jY_c \tan(\beta_g L)}{Y_c + jY_L \tan(\beta_g L)} = Y_L \bigg|_{\beta_g L = \pi}, \quad Y_c = Z_c^{-1}
\]  

(21.11)

if the line is loss-free. Below, the Smith charts illustrate the slot-impedance transformations and their addition, which produces a real normalized admittance, in three cases: (1) the patch is fed at one edge ($L_1 = 0$, $L_2 = L$), (2) the patch is fed at the center ($L_1 = L_2 = L / 2$), and (3) the patch is fed at a distance (feed inset) $z_0 = 0.165 \lambda$. 

Nikolova 2010
$L_2 = L \approx 0.46 \lambda$

$Y_m = Y_1 + Y_2 = 2 \times 0.09 = 0.18$

feed-point at the edge of cavity
$L \approx 0.46 \lambda$

$L_1 = L_2 = L/2 \approx 0.23 \lambda$

$Y_{in} = Y_1' + Y_2' = 2 \times 15 = 30$

feed-point at the middle of cavity
\[ Y_m = Y_1 + Y_2 = 2 \times 0.5 = 1 \]

\[ L_2 = L - z_0 \approx (0.46 - 0.165) \lambda = 0.315 \lambda \]

\[ L_1 = z_0 \approx 0.165 \lambda \]

\[ \lambda \approx \frac{c}{f} \]

\[ f \approx 1 \times 10^9 \text{ Hz} \]

\[ c = 3 \times 10^8 \text{ m/s} \]
The edge feed and the inset feed are illustrated below.

The two slots are separated by an electrical distance of $180^\circ$. However, because of the fringe effect the physical length $L$ of slightly less than $\lambda/2$. The reduction of the length is not much. Typically, it is $0.48\lambda \leq L \leq 0.49\lambda$.

Ideally, the resonant input impedance of the patch for the dominant TM$_{001}$ mode is entirely resistive and equal to half the transformed resistance of each slot:

$$Z_{in} = \frac{1}{Y_{in}} = \frac{1}{2G'_i} = R_{in}. \quad (21.12)$$

In reality, there is some mutual influence between the two slots, described by a mutual conductance and it should be included for more accurate calculations:

$$R_{in} = \frac{1}{2(G'_i \pm G_{12})}, \quad (21.13)$$

where the “+” sign relates to the odd modes, while the “−” sign relates to the even modes. Normally, $G_{12} \ll G'_i$.

For most patch antennas fed at the edge, $R_{in}$ is greater than the characteristic impedance $Z_c$ of the microstrip feed line (typically $Z_c = 50$ to $75 \ \Omega$). That is why, the inset-feed technique is widely used to achieve impedance match.
The figure below illustrates the normalized input impedance of a 1-D (along the y axis) loss-free open-ended transmission-line, whose behavior is very close to that of the dominant mode of the patch.

![Figure 14.14, pp 735, C. Balanis](image)

Using modal expansion, the input resistance for the inset-feed at $z = z_0$ is given approximately by

$$
R_{in} = \frac{1}{2(G_1 \pm G_{12})} \left[ \cos^2 \left( \frac{\pi}{L} z_0 \right) + \frac{G_1^2 + B_1^2}{Y_c^2} \sin^2 \left( \frac{\pi}{L} z_0 \right) - \frac{B_1}{Y_c} \sin \left( \frac{2\pi}{L} z_0 \right) \right].
$$

(21.14)

Here, $G_1$ and $B_1$ are calculated using (21.6) and (21.7). For most feeding microstrips, $G_1 / Y_c \ll 1$ and $B_1 / Y_c \ll 1$. Then,

$$
R_{in} = \frac{1}{2(G_1 \pm G_{12})} \cos^2 \left( \frac{\pi}{L} y_0 \right) = R_{in(z=0)} \cos^2 \left( \frac{\pi}{L} y_0 \right).
$$

(21.15)

Notice that the inset feeding technique for impedance match of the microstrip antennas is essentially identical to the off-center or asymmetrical feeding techniques for dipoles. In both cases, a position is sought along a resonant structure, where the current magnitude has the desired value.
2. Designing a rectangular patch using the transmission line model

Input data: $\varepsilon_r$, $h$, $f_r$

1) Calculate $W$ using (21.5).
2) Calculate $\varepsilon_{\text{reff}}$ using (21.5) and equation (6) from Lecture 20.
3) Calculate the extension $\Delta L$ due to the fringing effect using (21.1).
4) Calculate the actual (physical) length of the patch using

$$L = \frac{\lambda_0}{2} - 2\Delta L \quad \text{or} \quad L = \frac{1}{2f_r\sqrt{\varepsilon_{\text{reff}}\sqrt{\mu_0\varepsilon_0}}} - 2\Delta L. \quad (21.16)$$

5) Calculate radiating slot admittance using (21.6) and (21.7).
6) Calculate resonant input resistance at patch edge using (21.12) or (21.13) with $G' = G$ from (21.6).
7) If $R_{in}$ calculated in step 6 is too large, calculate the inset distance $z_0$ using (21.14) or (21.15).

3. Cavity model for the rectangular patch

The TL model is very limited in its description of the real processes taking place when a patch is excited. It takes into account only the $TM_{00n}$ modes where the energy propagates only in the longitudinal $z$ direction. The field distribution along the $x$ and $y$ axes is assumed uniform. It is true that the dominant $TM_{001}$ is prevalent but the performance of the patch is also affected by higher-order modes.

The cavity model is a more general model of the patch which imposes open-end conditions at the side edges of the patch. It represents the patch as a dielectric-loaded cavity with:

- electrical walls (above and below), and
- magnetic walls (around the perimeter of the patch).

The magnetic wall is a wall at which

$$\hat{n} \times \mathbf{H} = 0 \quad \text{(the $\mathbf{H}$-field is purely normal)}$$

$$\hat{n} \cdot \mathbf{E} = 0 \quad \text{(the $\mathbf{E}$-field is purely tangential)}$$

It is analogous to the open end termination in the theory of transmission lines.

If we treat the microstrip antenna only as a cavity, we can not represent...
radiation because an ideal loss-free cavity does not radiate and its input impedance is purely reactive. To account for the radiation, a loss mechanism is introduced. This is done by introducing an effective loss tangent, $\delta_{\text{eff}}$.

The thickness of the substrate is very small. The waves generated and propagating beneath the patch undergo considerable reflection at the edges of the patch. Only a very small fraction of them is being radiated. Thus, the antenna is quite inefficient. The cavity model assumes that the $E$ field is purely tangential to the slots formed between the ground plane and the patch edges (magnetic walls). Moreover, it considers only TM$^x$ modes, i.e., modes with no $H_x$ component. These assumptions are, basically, very much true.

The TM$^x$ modes are fully described by a single scalar function $A_x$ – the $x$-component of the magnetic vector potential:

$$
\mathbf{A} = A_x \hat{\mathbf{x}}. 
$$

(21.17)

In a homogeneous source-free medium, $A_x$ satisfies the wave equation:

$$
\nabla^2 A_x + k^2 A_x = 0.
$$

(21.18)

For regular shapes (like the rectangular cavity), it is advantageous to use the separation of variables:

$$
\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} + k^2 A_x = 0
$$

(21.19)

$$
A_x = X(x) Y(y) Z(z)
$$

(21.20)
The eigenvalue equation is

\[ k_x^2 + k_y^2 + k_z^2 = k^2. \]  

(21.23)

The solutions of (21.22) are harmonic functions:

\[ X(x) = \sum_{n} A_n^{c} \cos(k_{xn}x) + A_n^{s} \sin(k_{xn}x), \]

\[ Y(y) = \sum_{n} B_n^{c} \cos(k_{yn}y) + B_n^{s} \sin(k_{yn}y), \]  

(21.24)

\[ Z(z) = \sum_{n} C_n^{c} \cos(k_{zn}z) + C_n^{s} \sin(k_{zn}z). \]

When the functions in (21.24) are substituted in (21.20), they give the general solution of (21.18). The particular solution of (21.18) depends on the boundary conditions.

In our case, there are electric walls at \( x = 0 \) and \( x = h \). There, the tangential \( E \)-field components must vanish, i.e., \( E_y = E_z = 0 \) \( \mid_{x=0,h} \). Having in mind that

\[ E_x = \frac{1}{j \omega \mu \epsilon} \left( \frac{\partial^2 A_x}{\partial x^2} + k^2 A_x \right), \quad E_y = \frac{1}{j \omega \mu \epsilon} \left( \frac{\partial^2 A_x}{\partial x \partial y} \right), \quad E_z = \frac{1}{j \omega \mu \epsilon} \left( \frac{\partial^2 A_x}{\partial x \partial z} \right), \]  

(21.25)

we set \( A_x \) at the top and bottom walls as

\[ \frac{\partial A_x}{\partial x} \bigg|_{x=0,h} = 0. \]  

(21.26)

At all side walls, we set a vanishing normal derivative for \( A_x \):

\[ \frac{\partial A_x}{\partial z} \bigg|_{z=0,L} = 0, \quad \frac{\partial A_x}{\partial y} \bigg|_{y=0,W} = 0. \]  

(21.27)
This ensures vanishing $H_x$ and $H_y$ at $z=0$ and $z=L$, as well as vanishing $H_x$ and $H_z$ at $y=0$ and $y=W$ (magnetic walls), as follows from the relation between the H-field and $A_x$,

$$H_x = 0, \quad H_y = \frac{1}{\mu} \left( \frac{\partial A_x}{\partial z} \right), \quad H_x = \frac{1}{\mu} \left( \frac{\partial A_x}{\partial y} \right). \quad (21.28)$$

It is now obvious that the solution must appear in terms of the functions

$$X(x) = \sum_n A_n^x \cos(k_{xn}x), \quad k_{xn} = n \frac{\pi}{h},$$

$$Y(y) = \sum_n B_n^y \cos(k_{yn}y), \quad k_{yn} = n \frac{\pi}{W}, \quad \tag{21.29}$$

$$Z(z) = \sum_n C_n^z \cos(k_{zn}z), \quad k_{zn} = n \frac{\pi}{L}.$$  

The spectrum of the eigenmodes in the cavity is discrete. The frequencies of those modes (the resonant frequencies) can be calculated from (21.23) as

$$(\frac{m\pi}{h})^2 + (\frac{n\pi}{W})^2 + (\frac{p\pi}{L})^2 = (\omega_r^{(mnp)})^2 \mu \varepsilon, \quad (21.30)$$

$$f_r^{(mnp)} = \frac{1}{2\pi\sqrt{\mu \varepsilon}} \sqrt{\left(\frac{m\pi}{h}\right)^2 + \left(\frac{n\pi}{W}\right)^2 + \left(\frac{p\pi}{L}\right)^2}. \quad (21.31)$$

The mode with the lowest resonant frequency is the dominant mode. Since usually $L > W$, the lowest-frequency mode is the TM$^{x}_{001}$ mode, for which

$$f_r^{(001)} = \frac{1}{2\pi\sqrt{\mu \varepsilon}} \frac{\pi}{L} = \frac{c}{2L\sqrt{\varepsilon_r}}. \quad (21.32)$$

The dominant TM$^{x}_{001}$ mode is exactly the mode considered by the transmission-line model (see previous sections). The field distribution of some low-order modes is given in the following figure.
The general solution for the $A_{x}^{(mnp)}$ [see (21.20) and (21.24)] is

$$A_{x}^{(mnp)} = A_{x} \cos \left( \frac{m \pi}{h} x \right) B_{x} \cos \left( \frac{n \pi}{W} y \right) C_{x} \cos \left( \frac{p \pi}{L} z \right), \quad (21.33)$$

or

$$A_{x}^{(mnp)} = A_{mn} \cdot \cos \left( \frac{m \pi}{h} x \right) \cos \left( \frac{n \pi}{W} y \right) \cos \left( \frac{p \pi}{L} z \right). \quad (21.34)$$

The respective field solution for the $(m,n,p)$ mode is

$$E_{x} = -j \frac{\left( k_{x}^{2} - k_{y}^{2} \right)}{\omega \mu \varepsilon} A_{mn} \cdot \cos(k_{x}x) \cdot \cos(k_{y}y) \cdot \cos(k_{z}z), \quad (21.35)$$

$$E_{y} = -j \frac{k_{x}k_{y}}{\omega \mu \varepsilon} A_{mn} \cdot \sin(k_{x}x) \cdot \sin(k_{y}y) \cdot \cos(k_{z}z), \quad (21.36)$$
\[ E_z = -j \frac{k_x k_z}{\omega \mu \varepsilon} A_{mnp} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z), \quad (21.37) \]

\[ H_x = 0, \quad (21.38) \]

\[ H_y = -\frac{k_z}{\mu} A_{mnp} \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z), \quad (21.39) \]

\[ H_z = \frac{k_y}{\mu} A_{mnp} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z). \quad (21.40) \]

For the dominant TM\(_{001}^x\) mode,

\[ E_x = \left[ -j \left( k^2 - \pi^2 / h^2 \right) / (\omega \mu \varepsilon) \right] A_{001} \cos(\pi z / L), \quad E_y = E_z = 0, \quad (21.41) \]

\[ H_y = -(\pi / \mu L) A_{001} \sin(\pi z / L), \quad H_x = H_z = 0. \quad (21.42) \]

4. **Cavity model for the radiated field of a rectangular patch**

The microstrip patch is represented by the cavity model reasonably well assuming that the material of the substrate is truncated and does not extend beyond the edges of the patch. The four side walls (the magnetic walls) represent four narrow apertures (slots) through which radiation takes place.

The equivalence principle is used to calculate the radiation fields. The field inside the cavity is assumed equal to zero, and its influence on the field in the infinite region outside is represented by the equivalent surface currents on the surface of the cavity.

\[ \mathbf{J}_s, \mathbf{M}_s \]

\[ \mathbf{J}_s = \hat{n} \times \mathbf{H} \]

\[ \mathbf{M}_s = -\hat{n} \times \mathbf{E} \]
Because of the very small height $h$ of the substrate, the field is concentrated beneath the patch. There is some actual electrical current at the top metallic plate, however, its contribution to radiation is negligible. That is because: (1) it is backed by a conductor, and (2) it is very weak compared to the equivalent currents at the slots. The actual electrical current density of the top patch is maximum at the edges of the patch.

In the cavity model, the side walls employ magnetic-wall boundary condition, which sets the tangential $\mathbf{H}$ components at the slots equal to zero. Therefore,

$$\mathbf{J}_s = \hat{n} \times \mathbf{H} = 0. \quad (21.43)$$

Only the equivalent magnetic current density

$$\mathbf{M}_s = -\hat{n} \times \mathbf{E} \quad (21.44)$$

has substantial contribution to the radiated field.

The influence of the infinite ground plane is accounted for by the image theory, according to which the currents $\mathbf{M}_s$ in the presence of the infinite plane radiate as if magnetic currents of double strength radiate in free space:

$$\mathbf{M}_s = -2\hat{n} \times \mathbf{E}. \quad (21.45)$$

Note that an $E_x$ field at the slots corresponds to $\mathbf{M}_s$ density vector, which is tangential to the ground plane. Thus, its image is of the same direction. The equivalent magnetic current densities for the dominant $\text{TM}_{001}$ mode are sketched below.
At slots #1 and #2, the equivalent $M_s$ currents are co-directed and with equal amplitudes. They are constant along $x$ and $y$.

Radiation from a slot with constant current density

The radiation from an $(x-y)$ slot of constant $M_s$ currents is found using the electric vector potential $F$. Since $M_s$ has only a $y$ component, so does $F$: $F = F_y \hat{y}$.
\begin{equation}
F_y(r, \theta, \phi) = \frac{\varepsilon}{4\pi} \int_{-h/2}^{h/2} \int_{-W/2}^{W/2} \frac{M_y}{r_{PQ}} e^{-jk_0r_{PQ}} \, dx' \, dy' . \tag{21.46}
\end{equation}

Here, \( M_y = -2E_0 \), \( E_0 \) being the phasor of the \( E \)-field at the radiating slot, and \( r_{PQ} = r - r' \cdot \hat{r} = r - x' \sin \theta \cos \phi - y' \sin \theta \sin \phi \).

\begin{equation}
F_y = -2\varepsilon E_0 \frac{e^{-jk_0r}}{4\pi r} \int_{-h/2}^{h/2} \exp(jk_0x' \sin \theta \cos \phi) \, dx' \int_{-W/2}^{W/2} \exp(jk_0y' \sin \theta \sin \phi) \, dy' \tag{21.47}
\end{equation}

\begin{equation}
\Rightarrow F_y = -\frac{\varepsilon E_0 Wh}{2\pi r} \cdot e^{-jk_0r} \cdot \frac{\sin X}{X} \cdot \frac{\sin Y}{Y} \tag{21.48}
\end{equation}

where

\begin{align*}
X &= \frac{k_0h}{2} \sin \theta \cos \phi , \\
Y &= \frac{k_0W}{2} \sin \theta \sin \phi .
\end{align*}

According to the relation between the far-zone \( E \)-field and the vector potential,

\begin{equation}
E_r = 0, \quad E_\phi = j\omega \eta F_\theta , \quad E_\theta = -j\omega \eta F_\phi , \tag{21.49}
\end{equation}

where \( \eta = \sqrt{\mu_0 / \varepsilon_0} \), \( F_\theta = F_y \cos \theta \sin \phi \), and \( F_\phi = F_y \cos \phi \).

\begin{align*}
\Rightarrow E_\phi &= j\omega \varepsilon_0 \frac{WhE_0}{2\pi r} e^{-jk_0r} \cos \theta \sin \phi \frac{\sin X}{X} \frac{\sin Y}{Y} , \tag{21.50} \\
\Rightarrow E_\theta &= -j\omega \varepsilon_0 \frac{WhE_0}{2\pi r} e^{-jk_0r} \cos \phi \frac{\sin X}{X} \frac{\sin Y}{Y} . \tag{21.51}
\end{align*}

Since \( \omega \eta \varepsilon_0 = k_0 \),

\begin{align*}
E_\phi &= jk_0W \frac{V_0}{2\pi r} e^{-jk_0r} \left( \cos \theta \sin \phi \frac{\sin X}{X} \frac{\sin Y}{Y} \right) , \tag{21.52} \\
E_\theta &= -jk_0W \frac{V_0}{2\pi r} e^{-jk_0r} \left( \cos \phi \frac{\sin X}{X} \frac{\sin Y}{Y} \right) . \tag{21.53}
\end{align*}

Here \( V_0 = hE_0 \) is the voltage between the patch edge and the ground plane.

Slots #1 and #2 form an array of two elements with excitation of equal...
magnitude and phase, separated by the physical distance $L$. Their AF is

$$AF_{12} = 2\cos\left(\frac{k_0L_{\text{eff}}}{2}\cos\theta\right).$$

(21.54)

Here $L_{\text{eff}} = L + 2\Delta L$ is the effective patch length. Thus, the total radiation field is

$$E'_\phi = j\frac{k_0W_0}{\pi r}e^{-jk_0r}\left(\cos\theta\sin\phi\frac{\sin X \sin Y}{X \frac{Y}{\epsilon}}\right)\times \left[\cos\left(\frac{k_0L_{\text{eff}}}{2}\cos\theta\right)\right],$$

$$E'_\theta = -j\frac{k_0W_0}{\pi r}e^{-jk_0r}\left(\cos\phi\frac{\sin X \sin Y}{X \frac{Y}{\epsilon}}\right)\times \left[\cos\left(\frac{k_0L_{\text{eff}}}{2}\cos\theta\right)\right].$$

(21.55)

(21.56)

Introducing $Z = (k_0L_{\text{eff}} / 2)\cos\theta$, the pattern of the patch is obtained as

$$f(\theta,\phi) = \sqrt{E'^2_\phi + E'^2_\theta} = \sqrt{1 - \sin^2\phi \cdot \sin^2\theta \cdot \frac{\sin X \sin Y}{X \frac{Y}{\epsilon}}\cos Z}.$$  

(21.57)

E-plane pattern (xz plane, $\psi = 0^\circ$, $0^\circ \leq \theta \leq 180^\circ$)

$$f_E(\theta) = \frac{\sin\left(\frac{k_0h\sin\theta}{2}\right)}{\frac{k_0h}{2}\sin\theta} \cdot \cos\left(\frac{k_0L_{\text{eff}}}{2}\cos\theta\right).$$

(21.58)
**H-plane pattern** (xy plane, $\theta = 90^\circ$, $0^\circ \leq \phi \leq 90^\circ$ and $270^\circ \leq \phi \leq 360^\circ$)

\[ f_H(\theta) = \cos \phi \cdot \sin \left( \frac{k_0h}{2} \cos \phi \right) \sin \left( \frac{k_0W}{2} \sin \phi \right) \]

(21.59)

\[ k_0h = 0.1; \quad \varepsilon_r = 4.0 \]

\[ L_{\text{eff}} = 0.5\lambda_0 / \sqrt{\varepsilon_r} \]
$k_0 h = 0.1; \varepsilon_r = 4.0$

$W = 0.5 \lambda_0 / \sqrt{\varepsilon_r}$
Fig. 14.17, p. 746, Balanis

(a) \( E \)-plane \((\theta = 90^\circ)\)

- Measured
- Moment method (Courtesy D. Pozar)
- Cavity model
Non-radiating slots: It can be shown that the slots at $y = -W/2$ and $y = W/2$ do no radiate in the principle E- and H-planes. In general, these two slots do radiate away from the principle planes, but their field intensity is everywhere small compared to that radiated by slots #1 and #2.