The Heaviside Step Function − Numerical Implementation and Applicaitons

1. Introduction

The unit step function or the Heaviside step function introduced by the great English scientist Oliver Heaviside [1] are frequently used in mathematics, science and engineering [2]-[7]. The step function can be approximated by many forms of the analytic expression [8]-[10], where the slope becomes infinite as a controlling parameter in the equation approaches zero or infinity [11]. Most of existing literatures on the numerically stable expression for the step function have an arbitrary uncontolled constant for the step transition. In [11], two expressions are given for the numerical approximation of the step function. This paper extends the result of [11], to arrive at a more compact form for the numerical calculation of the step function. The expression has only one parameter for controllinig the rate of a step transition. Expressions for the pulse, ramp, doublet, and delta functions from the step function are derived from the step function representation.

2. The Heaviside Step Function

The Heaviside step function is defined in a following piecewise form [1].

(1) $$H(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ 1, & x > 0 \end{cases}$$

Among many analytic approximations to the Heaviside step function [8], the following logistic function

(2) $$H(x) = \lim_{a \to \infty} \frac{1}{1 + e^{-ax}}$$

and the one involving the hyperbolic tangent function

(3) $$H(x) = \lim_{a \to \infty} \frac{1}{2}\left(1 + \tanh ax\right)$$

are most convenient for numerical programming [11].

An algebraic manipulation shows that (2) and (3) are same. The constant a in (2) and (3) controls the slope of the step transition, which is usually set to an arbitrarily large value. The exponential term e^{-ax} in (2) causes an overflow error when ax is a large negative number, while this does not happen in (3) due to the characteristics of the programming language's internal routine for the hyperbolic tangent function [11].

Fig. 1 shows (3) for the case when a is 2, 8 and the infinity [5]. There is an arbitrariness in the choice of the slope parameter a and this work presents an expression for a controlled transition of the step function.

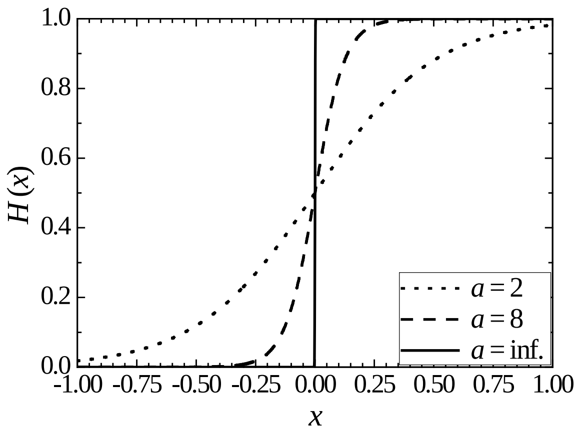

Fig. 1 Approximations to the Heaviside unit step function [11].

The following expression is given in [11] for the numerical expression for the step function.

(4) $$H(x-x_0)=\frac{1}{2}\left[1+\tanh\frac{b(x-x_0)}{\Delta}\right]$$

The parameter b in (4) is determined from the following condition:

(5) $$H(x-x_0)=\begin{cases}\Delta, & x=x_0-\Delta\\ 1/2, & x_0-\Delta\le x\le x_0+\Delta\\ 1-\Delta, & x=x_0+\Delta\end{cases}$$

where Δ is an arbitrarily small number that can be represented in a computing machine used for numericall calculation. The constant can be obtained from (5), *viz.*,

(6) $$\frac{1+\tanh b}{2}=1-\Delta\to b=\tanh^{-1}(1-2\Delta)=\frac{1}{2}\ln\frac{2-2\Delta}{2\Delta}\simeq-\frac{1}{2}\ln\Delta$$

so that the constant a in (3) is now given by

(7a) $$\boxed{a=-\frac{1}{2\Delta}\ln\Delta}$$

(7b) $$\boxed{H_i=H(x-x_i)=\frac{1}{2}[1+\tanh a(x-x_i)]}$$

The smallest practical value of Δ is 10^{-8} in a single-precision Fortran and 10^{-16} in a double-precision arithmetic. The corresponding values of a are 8×10^{8} and 17×10^{16}, respectively.

Expressions for the the delta function, the doublet function, and the ramp function are derived from (3) as follows.

(8) $$\boxed{\delta(x)\equiv H'(x)=\frac{a}{2}(1-\tanh^2 ax)}$$

(9) $$\boxed{d(x)=H''(x)=-a^2(1-\tanh^2 ax)\tanh ax}$$

(10) $$\boxed{r(x)=\int H(x)\,dx=\frac{1}{2}\left(x+\frac{1}{a}\ln\cosh ax\right)=\frac{1}{2}\left(x+|x|+\frac{1}{a}\ln\frac{1+e^{-2a|x|}}{2}\right)}$$

The equation (3) can also be used to form a pulse function that takes a value of unity at $x_1\le x\le x_2$:

(11) $$\boxed{\Pi_{12}=\Pi(x;x_1,x_2)=H(x-x_1)-H(x-x_2)=\frac{1}{2}[\tanh a(x-x_1)-\tanh a(x-x_2)]}$$

The negative unit step function:

(12) $$\boxed{L_i\equiv1-H(x-x_i)=\frac{1}{2}[1-\tanh a(x-x_i)]}$$

Note that equations (8)–(11), in the entire range $-\infty<x<\infty$, do not possess terms that cause the numerical overflow problem. The transition-controlling constant a given by (7) is applicable to all of equations (8) to (11).

3. Applications

1) Piecewisely defincd function

(13a) $$f(x)=\begin{cases}f_1(x), x<x_1\\ f_{12}(x), x_1\le x\le x_2\\ f_2(x), x>x_2\end{cases}$$

(13b) $f(x) = f_1(x)L_1(x) + f_{12}(x)\Pi_{12}(x) + f_2(x)H_2(x)$

References

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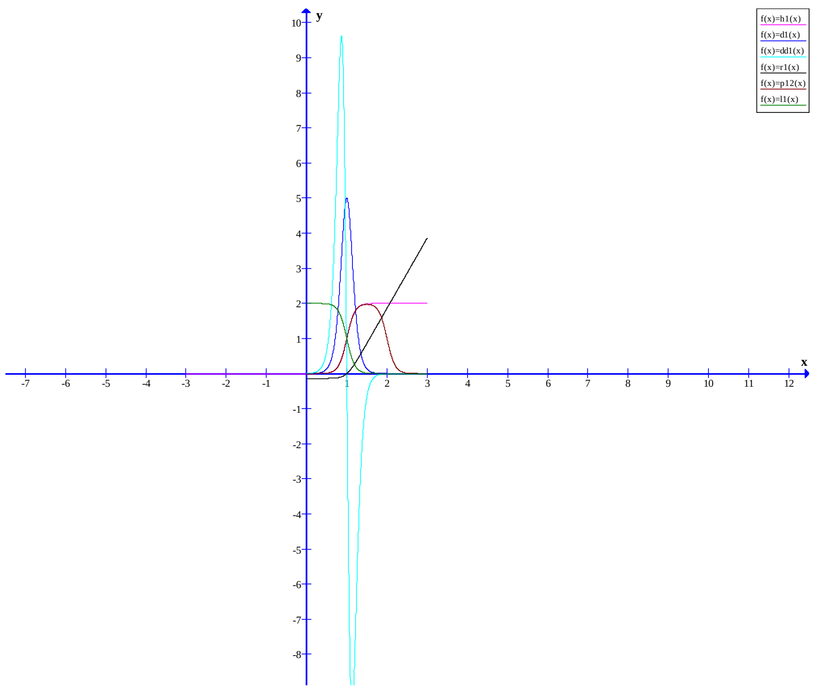
Appendix:

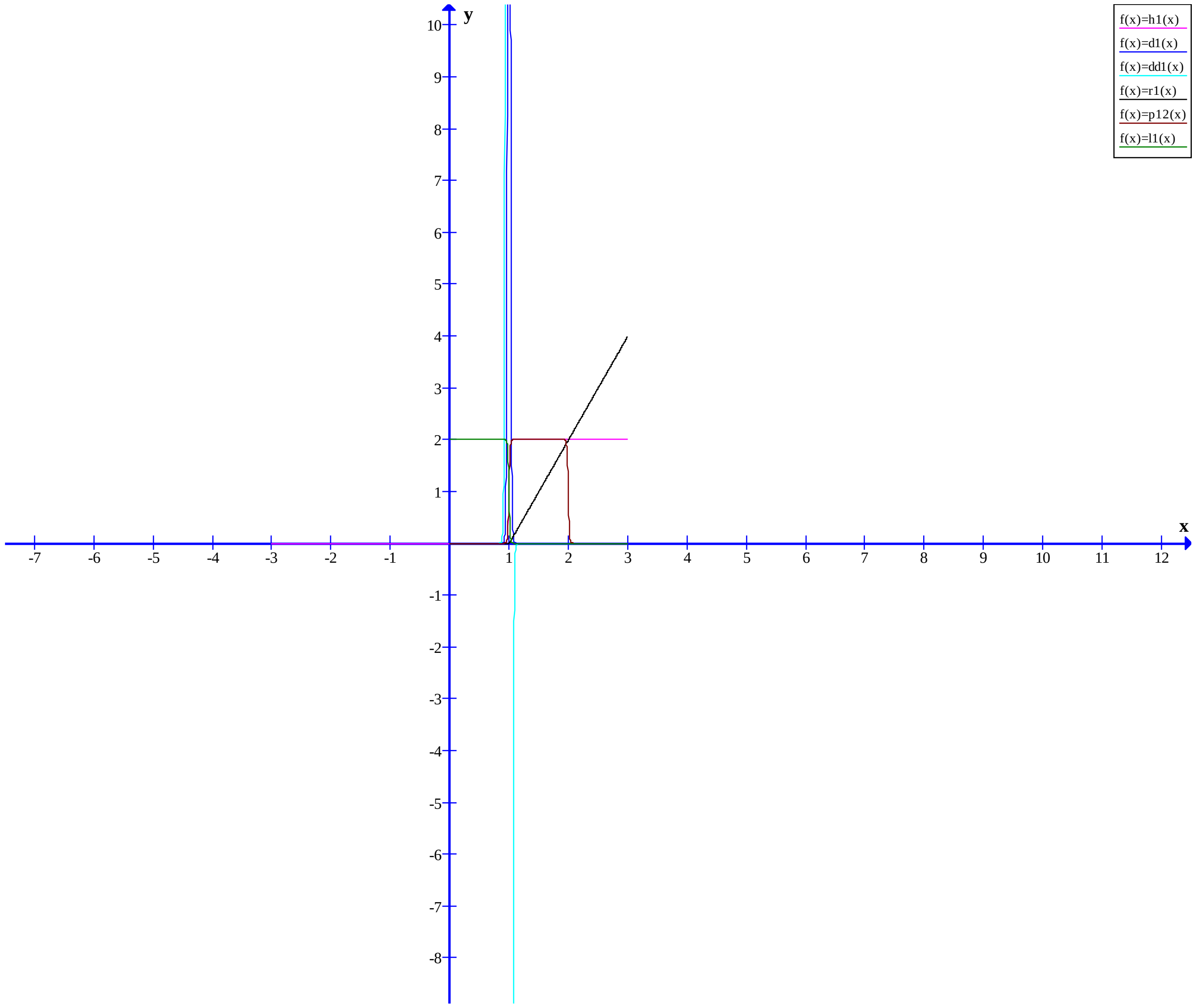
Implementation example

The following graphs have been obtained using 'Graph v4.4.2'

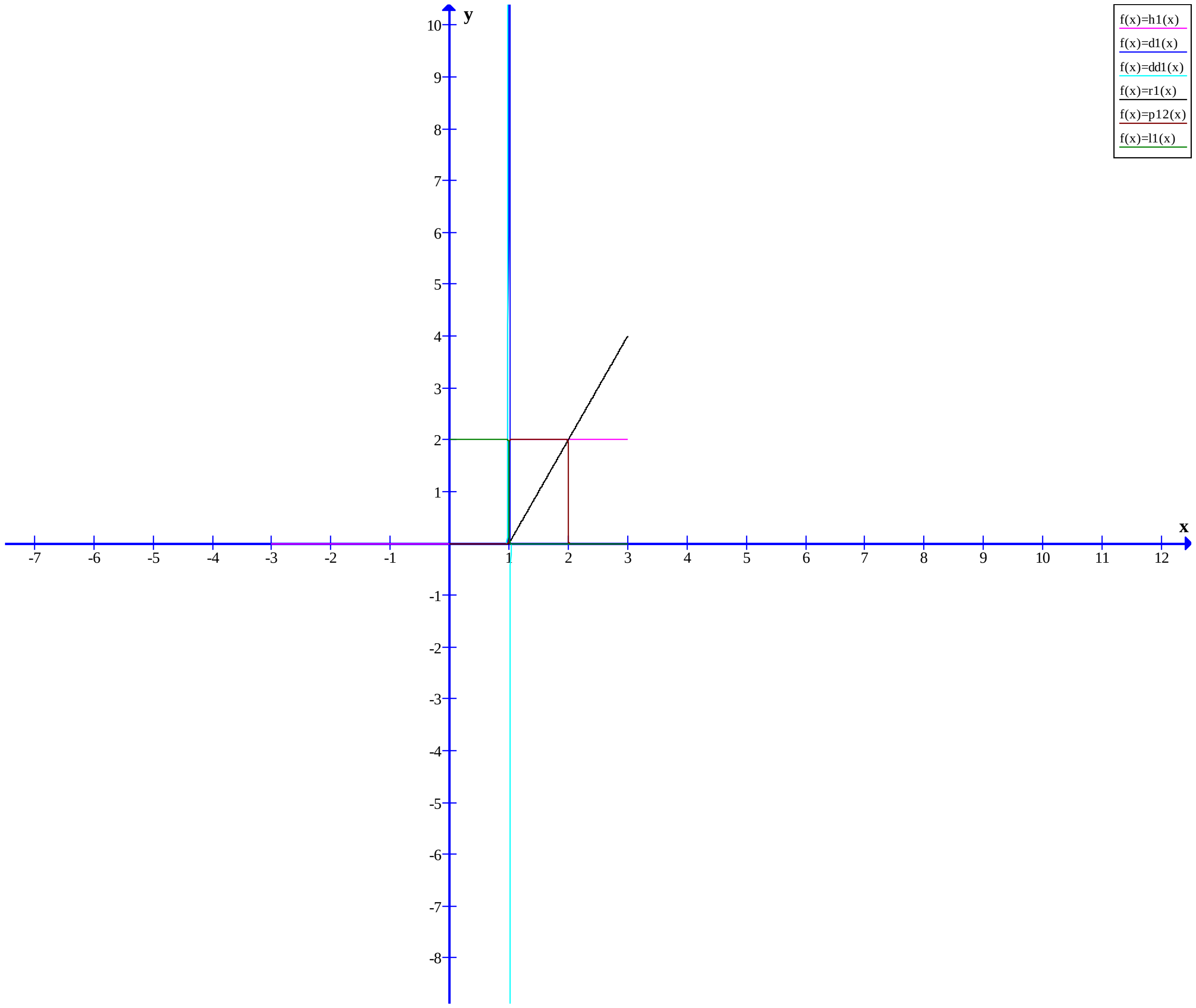
$a = 5$



$a = 50$



$a = 230$



y
x
f(x)=h1(x)
f(x)=d1(x)
f(x)=dd1(x)
f(x)=r1(x)
f(x)=p12(x)
f(x)=l1(x)