# Chapter 12

# WAVEGUIDES

If a man writes a better book, preaches a better sermon, or makes a better mouse-trap than his neighbor, the world will make a beaten path to his door.

—RALPH WALDO EMERSON

## 12.1 INTRODUCTION

As mentioned in the preceding chapter, a transmission line can be used to guide EM energy from one point (generator) to another (load). A waveguide is another means of achieving the same goal. However, a waveguide differs from a transmission line in some respects, although we may regard the latter as a special case of the former. In the first place, a transmission line can support only a transverse electromagnetic (TEM) wave, whereas a waveguide can support many possible field configurations. Second, at microwave frequencies (roughly 3–300 GHz), transmission lines become inefficient due to skin effect and dielectric losses; waveguides are used at that range of frequencies to obtain larger bandwidth and lower signal attenuation. Moreover, a transmission line may operate from dc ($f = 0$) to a very high frequency; a waveguide can operate only above a certain frequency called the *cutoff frequency* and therefore acts as a high-pass filter. Thus, waveguides cannot transmit dc, and they become excessively large at frequencies below microwave frequencies.

Although a waveguide may assume any arbitrary but uniform cross section, common waveguides are either rectangular or circular. Typical waveguides[1] are shown in Figure 12.1. Analysis of circular waveguides is involved and requires familiarity with Bessel functions, which are beyond our scope.[2] We will consider only rectangular waveguides. By assuming lossless waveguides ($\sigma_c \simeq \infty$, $\sigma \approx 0$), we shall apply Maxwell's equations with the appropriate boundary conditions to obtain different modes of wave propagation and the corresponding **E** and **H** fields.

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[1]For other types of waveguides, see J. A. Seeger, *Microwave Theory, Components and Devices.* Englewood Cliffs, NJ: Prentice-Hall, 1986, pp. 128–133.

[2]Analysis of circular waveguides can be found in advanced EM or EM-related texts, e.g., S. Y. Liao. *Microwave Devices and Circuits,* 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1990, pp. 119–141.

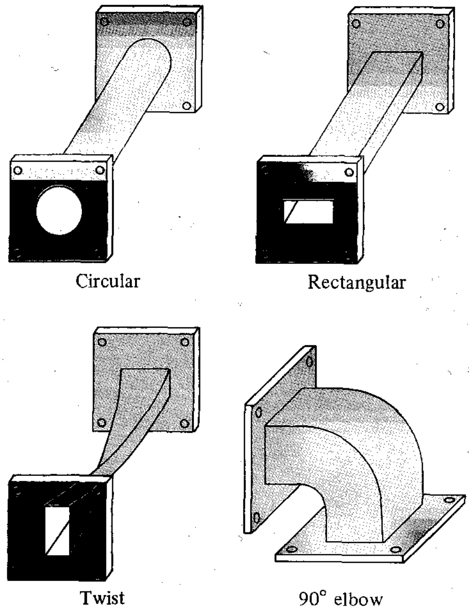


**Figure 12.1** Typical waveguides.

## 12.2 RECTANGULAR WAVEGUIDES

Consider the rectangular waveguide shown in Figure 12.2. We shall assume that the waveguide is filled with a source-free ($\rho_v = 0, \mathbf{J} = 0$) lossless dielectric material ($\sigma \simeq 0$) and its walls are perfectly conducting ($\sigma_c \simeq \infty$). From eqs. (10.17) and (10.19), we recall that for a lossless medium, Maxwell's equations in phasor form become

$$\nabla^2 \mathbf{E}_s + k^2 \mathbf{E}_s = 0 \tag{12.1}$$

$$\nabla^2 \mathbf{H}_s + k^2 \mathbf{H}_s = 0 \tag{12.2}$$

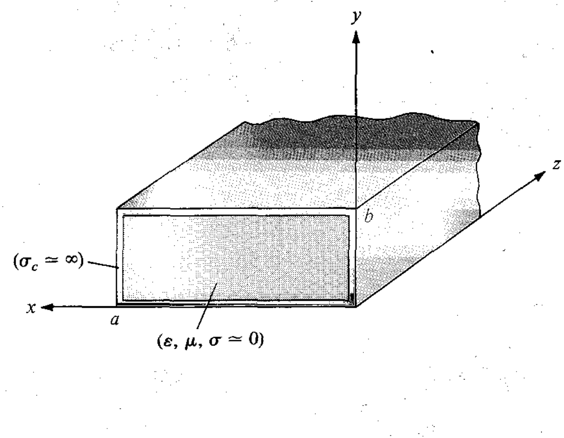


**Figure 12.2** A rectangular waveguide with perfectly conducting walls, filled with a lossless material.

where

$$\boxed{k = \omega\sqrt{\mu\varepsilon}} \tag{12.3}$$

and the time factor $e^{j\omega t}$ is assumed. If we let

$$\mathbf{E}_s = (E_{xs}, E_{ys}, E_{zs}) \qquad \text{and} \qquad \mathbf{H}_s = (H_{xs}, H_{ys}, H_{zs})$$

each of eqs. (12.1) and (12.2) is comprised of three scalar Helmholtz equations. In other words, to obtain **E** and **H** fields, we have to solve six scalar equations. For the $z$-component, for example, eq. (12.1) becomes

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} + k^2 E_{zs} = 0 \tag{12.4}$$

which is a partial differential equation. From Example 6.5, we know that eq. (12.4) can be solved by separation of variables (product solution). So we let

$$E_{zs}(x, y, z) = X(x)\, Y(y)\, Z(z) \tag{12.5}$$

where $X(x)$, $Y(y)$, and $Z(z)$ are functions of $x$, $y$, and $z$, respectively. Substituting eq. (12.5) into eq. (12.4) and dividing by $XYZ$ gives

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2 \tag{12.6}$$

Since the variables are independent, each term in eq. (12.6) must be constant, so the equation can be written as

$$-k_x^2 - k_y^2 + \gamma^2 = -k^2 \tag{12.7}$$

where $-k_x^2$, $-k_y^2$, and $\gamma^2$ are separation constants. Thus, eq. (12.6) is separated as

$$X'' + k_x^2 X = 0 \tag{12.8a}$$

$$Y'' + k_y^2 Y = 0 \tag{12.8b}$$

$$Z'' - \gamma^2 Z = 0 \tag{12.8c}$$

By following the same argument as in Example 6.5, we obtain the solution to eq. (12.8) as

$$X(x) = c_1 \cos k_x x + c_2 \sin k_x x \tag{12.9a}$$

$$Y(y) = c_3 \cos k_y y + c_4 \sin k_y y \tag{12.9b}$$

$$Z(z) = c_5 e^{\gamma z} + c_6 e^{-\gamma z} \tag{12.9c}$$

Substituting eq. (12.9) into eq. (12.5) gives

$$\begin{aligned} E_{zs}(x, y, z) = {} & (c_1 \cos k_x x + c_2 \sin k_x x)(c_3 \cos k_y y \\ & + c_4 \sin k_y y)\,(c_5 e^{\gamma z} + c_6 e^{-\gamma z}) \end{aligned} \tag{12.10}$$

As usual, if we assume that the wave propagates along the waveguide in the $+z$-direction, the multiplicative constant $c_5 = 0$ because the wave has to be finite at infinity [i.e., $E_{zs}(x, y, z = \infty) = 0$]. Hence eq. (12.10) is reduced to

$$E_{zs}(x, y, z) = (A_1 \cos k_x x + A_2 \sin k_x x)(A_3 \cos k_y y + A_4 \sin k_y y)e^{-\gamma z} \tag{12.11}$$

where $A_1 = c_1c_6$, $A_2 = c_2c_6$, and so on. By taking similar steps, we get the solution of the $z$-component of eq. (12.2) as

$$H_{zs}(x, y, z) = (B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)e^{-\gamma z} \tag{12.12}$$

Instead of solving for other field component $E_{xs}$, $E_{ys}$, $H_{xs}$, and $H_{ys}$ in eqs. (12.1) and (12.2) in the same manner, we simply use Maxwell's equations to determine them from $E_{zs}$ and $H_{zs}$. From

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s$$

and

$$\nabla \times \mathbf{H}_s = j\omega\varepsilon\mathbf{E}_s$$

we obtain

$$\frac{\partial E_{zs}}{\partial y} - \frac{\partial E_{ys}}{\partial z} = -j\omega\mu H_{xs} \tag{12.13a}$$

$$\frac{\partial H_{zs}}{\partial y} - \frac{\partial H_{ys}}{\partial z} = j\omega\varepsilon E_{xs} \tag{12.13b}$$

$$\frac{\partial E_{xs}}{\partial z} - \frac{\partial E_{zs}}{\partial x} = j\omega\mu H_{ys} \tag{12.13c}$$

$$\frac{\partial H_{xs}}{\partial z} - \frac{\partial H_{zs}}{\partial x} = j\omega\varepsilon E_{ys} \tag{12.13d}$$

$$\frac{\partial E_{ys}}{\partial x} - \frac{\partial E_{xs}}{\partial y} = -j\omega\mu H_{zs} \tag{12.13e}$$

$$\frac{\partial H_{ys}}{\partial x} - \frac{\partial H_{xs}}{\partial y} = j\omega\varepsilon E_{zs} \tag{12.13f}$$

We will now express $E_{xs}$, $E_{ys}$, $H_{xs}$, and $H_{ys}$ in terms of $E_{zs}$ and $H_{zs}$. For $E_{xs}$, for example, we combine eqs. (12.13b) and (12.13c) and obtain

$$j\omega\varepsilon E_{xs} = \frac{\partial H_{zs}}{\partial y} + \frac{1}{j\omega\mu}\left(\frac{\partial^2 E_{xs}}{\partial z^2} - \frac{\partial^2 E_{zs}}{\partial x \partial z}\right) \tag{12.14}$$

From eqs. (12.11) and (12.12), it is clear that all field components vary with $z$ according to $e^{-\gamma z}$, that is,

$$E_{zs} \sim e^{-\gamma z}, \qquad E_{xs} \sim e^{-\gamma z}$$

Hence

$$\frac{\partial E_{zs}}{\partial z} = -\gamma E_{zs}, \qquad \frac{\partial^2 E_{xx}}{\partial z^2} = \gamma^2 E_{xs}$$

and eq. (12.14) becomes

$$j\omega\varepsilon E_{xs} = \frac{\partial H_{zs}}{\partial y} + \frac{1}{j\omega\mu}\left(\gamma^2 E_{xs} + \gamma \frac{\partial E_{zs}}{\partial x}\right)$$

or

$$-\frac{1}{j\omega\mu}(\gamma^2 + \omega^2\mu\varepsilon) E_{xs} = \frac{\gamma}{j\omega\mu}\frac{\partial E_{zs}}{\partial x} + \frac{\partial H_{zs}}{\partial y}$$

Thus, if we let $h^2 = \gamma^2 + \omega^2\mu\varepsilon = \gamma^2 + k^2$,

$$E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y}$$

Similar manipulations of eq. (12.13) yield expressions for $E_{ys}$, $H_{xs}$, and $H_{ys}$ in terms of $E_{zs}$ and $H_{zs}$. Thus,

$$E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y} \tag{12.15a}$$

$$E_{ys} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial x} \tag{12.15b}$$

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial x} \tag{12.15c}$$

$$H_{ys} = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial y} \tag{12.15d}$$

where

$$h^2 = \gamma^2 + k^2 = k_x^2 + k_y^2 \tag{12.16}$$

Thus we can use eq. (12.15) in conjunction with eqs. (12.11) and (12.12) to obtain $E_{xs}$, $E_{ys}$, $H_{xs}$, and $H_{ys}$.

From eqs. (12.11), (12.12), and (12.15), we notice that there are different types of field patterns or configurations. Each of these distinct field patterns is called a *mode*. Four different mode categories can exist, namely:

1. $E_{zs} = 0 = H_{zs}$ (TEM mode): This is the *transverse electromagnetic* (TEM) mode, in which both the **E** and **H** fields are transverse to the direction of wave propagation. From eq. (12.15), all field components vanish for $E_{zs} = 0 = H_{zs}$. Consequently, we conclude that a rectangular waveguide cannot support TEM mode.

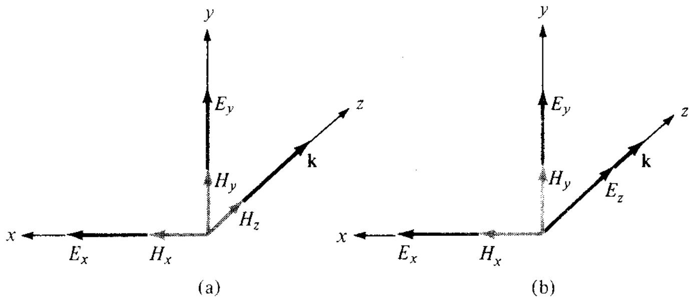


Figure 12.3 Components of EM fields in a rectangular waveguide: (a) TE mode $E_z = 0$, (b) TM mode, $H_z = 0$.

2. $E_{zs} = 0$, $H_{zs} \neq 0$ (TE modes): For this case, the remaining components ($E_{xs}$ and $E_{ys}$) of the electric field are transverse to the direction of propagation $\mathbf{a}_z$. Under this condition, fields are said to be in *transverse electric* (TE) modes. See Figure 12.3(a).
3. $E_{zs} \neq 0$, $H_{zs} = 0$ (TM modes): In this case, the **H** field is transverse to the direction of wave propagation. Thus we have *transverse magnetic* (TM) modes. See Figure 12.3(b).
4. $E_{zs} \neq 0$, $H_{zs} \neq 0$ (HE modes): This is the case when neither **E** nor **H** field is transverse to the direction of wave propagation. They are sometimes referred to as *hybrid* modes.

We should note the relationship between $k$ in eq. (12.3) and $\beta$ of eq. (10.43a). The phase constant $\beta$ in eq. (10.43a) was derived for TEM mode. For the TEM mode, $h = 0$, so from eq. (12.16), $\gamma^2 = -k^2 \rightarrow \gamma = \alpha + j\beta = jk$; that is, $\beta = k$. For other modes, $\beta \neq k$. In the subsequent sections, we shall examine the TM and TE modes of propagation separately.

## 12.3 TRANSVERSE MAGNETIC (TM) MODES

For this case, the magnetic field has its components transverse (or normal) to the direction of wave propagation. This implies that we set $H_z = 0$ and determine $E_x$, $E_y$, $E_z$, $H_x$, and $H_y$ using eqs. (12.11) and (12.15) and the boundary conditions. We shall solve for $E_z$ and later determine other field components from $E_z$. At the walls of the waveguide, the tangential components of the **E** field must be continuous; that is,

$$E_{zs} = 0 \quad \text{at} \quad y = 0 \tag{12.17a}$$

$$E_{zs} = 0 \quad \text{at} \quad y = b \tag{12.17b}$$

$$E_{zs} = 0 \quad \text{at} \quad x = 0 \tag{12.17c}$$

$$E_{zs} = 0 \quad \text{at} \quad x = a \tag{12.17d}$$

Equations (12.17a) and (12.17c) require that $A_1 = 0 = A_3$ in eq. (12.11), so eq. (12.11) becomes

$$E_{zs} = E_o \sin k_x x \sin k_y y \, e^{-\gamma z} \tag{12.18}$$

where $E_o = A_2 A_4$. Also eqs. (12.17b) and (12.17d) when applied to eq. (12.18) require that

$$\sin k_x a = 0, \qquad \sin k_y b = 0 \tag{12.19}$$

This implies that

$$k_x a = m\pi, \qquad m = 1, 2, 3, \ldots \tag{12.20a}$$

$$k_y b = n\pi, \qquad n = 1, 2, 3, \ldots \tag{12.20b}$$

or

$$k_x = \frac{m\pi}{a}, \qquad k_y = \frac{n\pi}{b} \tag{12.21}$$

The negative integers are not chosen for $m$ and $n$ in eq. (12.20a) for the reason given in Example 6.5. Substituting eq. (12.21) into eq. (12.18) gives

$$E_{zs} = E_o \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{12.22}$$

We obtain other field components from eqs. (12.22) and (12.15) bearing in mind that $H_{zs} = 0$. Thus

$$E_{xs} = -\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) E_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{12.23a}$$

$$E_{ys} = -\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right) E_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{12.23b}$$

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right) E_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{12.23c}$$

$$H_{ys} = -\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right) E_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{12.23d}$$

where

$$h^2 = k_x^2 + k_y^2 = \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2 \tag{12.24}$$

which is obtained from eqs. (12.16) and (12.21). Notice from eqs. (12.22) and (12.23) that each set of integers $m$ and $n$ gives a different field pattern or mode, referred to as $TM_{mn}$

mode, in the waveguide. Integer $m$ equals the number of half-cycle variations in the $x$-direction, and integer $n$ is the number of half-cycle variations in the $y$-direction. We also notice from eqs. (12.22) and (12.23) that if $(m, n)$ is $(0, 0)$, $(0, n)$, or $(m, 0)$, all field components vanish. Thus neither $m$ nor $n$ can be zero. Consequently, $TM_{11}$ is the lowest-order mode of all the $TM_{mn}$ modes.

By substituting eq. (12.21) into eq. (12.16), we obtain the propagation constant

$$\gamma = \sqrt{\left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2 - k^2} \tag{12.25}$$

where $k = \omega\sqrt{\mu\varepsilon}$ as in eq. (12.3). We recall that, in general, $\gamma = \alpha + j\beta$. In the case of eq. (12.25), we have three possibilities depending on $k$ (or $\omega$), $m$, and $n$:

**CASE A (cutoff):**

If

$$k^2 = \omega^2\mu\varepsilon = \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2$$

$$\gamma = 0 \qquad \text{or} \qquad \alpha = 0 = \beta$$

The value of $\omega$ that causes this is called the *cutoff angular frequency* $\omega_c$; that is,

$$\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2} \tag{12.26}$$

**CASE B (evanescent):**

If

$$k^2 = \omega^2\mu\varepsilon < \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2$$

$$\gamma = \alpha, \qquad \beta = 0$$

In this case, we have no wave propagation at all. These nonpropagating or attenuating modes are said to be *evanescent.*

**CASE C (propagation):**

If

$$k^2 = \omega^2\mu\varepsilon > \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2$$

$$\gamma = j\beta, \qquad \alpha = 0$$

that is, from eq. (12.25) the phase constant $\beta$ becomes

$$\beta = \sqrt{k^2 - \left[\frac{m\pi}{a}\right]^2 - \left[\frac{n\pi}{b}\right]^2} \tag{12.27}$$

This is the only case when propagation takes place because all field components will have the factor $e^{-\gamma z} = e^{-j\beta z}$.

Thus for each mode, characterized by a set of integers $m$ and $n$, there is a corresponding *cutoff frequency* $f_c$

The **cutoff frequency** is the operating frequency below which attenuation occurs and above which propagation takes place.

The waveguide therefore operates as a high-pass filter. The cutoff frequency is obtained from eq. (12.26) as

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2}$$

or

$$\boxed{f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \tag{12.28}$$

where $u' = \dfrac{1}{\sqrt{\mu\varepsilon}}$ = phase velocity of uniform plane wave in the lossless dielectric medium ($\sigma = 0, \mu, \varepsilon$) filling the waveguide. The *cutoff wavelength* $\lambda_c$ is given by

$$\lambda_c = \frac{u'}{f_c}$$

or

$$\boxed{\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}} \tag{12.29}$$

Note from eqs. (12.28) and (12.29) that $TM_{11}$ has the lowest cutoff frequency (or the longest cutoff wavelength) of all the TM modes. The phase constant $\beta$ in eq. (12.27) can be written in terms of $f_c$ as

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left[\frac{f_c}{f}\right]^2}$$

or

$$\beta = \beta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2} \tag{12.30}$$

where $\beta' = \omega/u' = \omega\sqrt{\mu\varepsilon}$ = phase constant of uniform plane wave in the dielectric medium. It should be noted that $\gamma$ for evanescent mode can be expressed in terms of $f_c$, namely,

$$\gamma = \alpha = \beta' \sqrt{\left(\frac{f_c}{f}\right)^2 - 1} \tag{12.30a}$$

The phase velocity $u_p$ and the wavelength in the guide are, respectively, given by

$$u_p = \frac{\omega}{\beta}, \qquad \lambda = \frac{2\pi}{\beta} = \frac{u_p}{f} \tag{12.31}$$

The intrinsic wave impedance of the mode is obtained from eq. (12.23) as ($\gamma = j\beta$)

$$\begin{aligned}\eta_{TM} &= \frac{E_x}{H_y} = -\frac{E_y}{H_x} \\ &= \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} \sqrt{1 - \left[\frac{f_c}{f}\right]^2}\end{aligned}$$

or

$$\eta_{TM} = \eta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2} \tag{12.32}$$

where $\eta' = \sqrt{\mu/\varepsilon}$ = intrinsic impedance of uniform plane wave in the medium. Note the difference between $u'$, $\beta'$, and $\eta'$, and $u$, $\beta$, and $\eta$. The quantities with prime are wave characteristics of the dielectric medium unbounded by the waveguide as discussed in Chapter 10 (i.e., for TEM mode). For example, $u'$ would be the velocity of the wave if the waveguide were removed and the entire space were filled with the dielectric. The quantities without prime are the wave characteristics of the medium bounded by the waveguide.

As mentioned before, the integers $m$ and $n$ indicate the number of half-cycle variations in the $x$–$y$ cross section of the guide. Thus for a fixed time, the field configuration of Figure 12.4 results for $TM_{21}$ mode, for example.

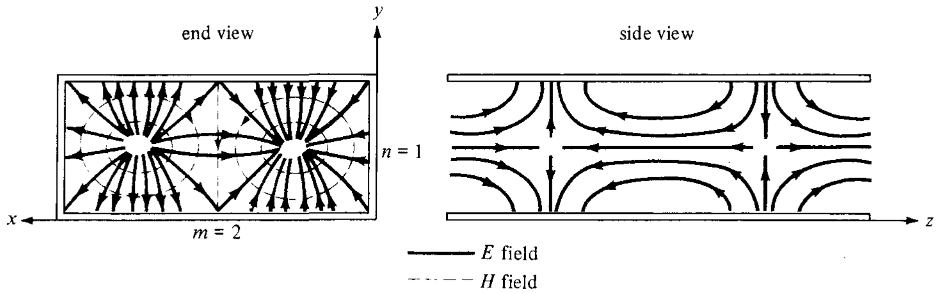


**Figure 12.4** Field configuration for $TM_{21}$ mode.

## 12.4 TRANSVERSE ELECTRIC (TE) MODES

In the TE modes, the electric field is transverse (or normal) to the direction of wave propagation. We set $E_z = 0$ and determine other field components $E_x$, $E_y$, $H_x$, $H_y$, and $H_z$ from eqs. (12.12) and (12.15) and the boundary conditions just as we did for the TM modes. The boundary conditions are obtained from the fact that the tangential components of the electric field must be continuous at the walls of the waveguide; that is,

$$E_{xs} = 0 \quad \text{at} \quad y = 0 \tag{12.33a}$$

$$E_{xs} = 0 \quad \text{at} \quad y = b \tag{12.33b}$$

$$E_{ys} = 0 \quad \text{at} \quad x = 0 \tag{12.33c}$$

$$E_{ys} = 0 \quad \text{at} \quad x = a \tag{12.33d}$$

From eqs. (12.15) and (12.33), the boundary conditions can be written as

$$\frac{\partial H_{zs}}{\partial y} = 0 \quad \text{at} \quad y = 0 \tag{12.34a}$$

$$\frac{\partial H_{zs}}{\partial y} = 0 \quad \text{at} \quad y = b \tag{12.34b}$$

$$\frac{\partial H_{zs}}{\partial x} = 0 \quad \text{at} \quad x = 0 \tag{12.34c}$$

$$\frac{\partial H_{zs}}{\partial x} = 0 \quad \text{at} \quad x = a \tag{12.34d}$$

Imposing these boundary conditions on eq. (12.12) yields

$$\boxed{H_{zs} = H_o \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z}} \tag{12.35}$$

where $H_o = B_1B_3$. Other field components are easily obtained from eqs. (12.35) and (12.15) as

$$E_{xs} = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right) H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{12.36a}$$

$$E_{ys} = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right) H_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{12.36b}$$

$$H_{xs} = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) H_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{12.36c}$$

$$H_{ys} = \frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right) H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{12.36d}$$

where $m = 0, 1, 2, 3, \ldots$; and $n = 0, 1, 2, 3, \ldots$; $h$ and $\gamma$ remain as defined for the TM modes. Again, $m$ and $n$ denote the number of half-cycle variations in the $x$–$y$ cross section of the guide. For $TE_{32}$ mode, for example, the field configuration is in Figure 12.5. The cutoff frequency $f_c$, the cutoff wavelength $\lambda_c$, the phase constant $\beta$, the phase velocity $u_p$, and the wavelength $\lambda$ for TE modes are the same as for TM modes [see eqs. (12.28) to (12.31)].

For TE modes, $(m, n)$ may be $(0, 1)$ or $(1, 0)$ but not $(0, 0)$. Both $m$ and $n$ cannot be zero at the same time because this will force the field components in eq. (12.36) to vanish. This implies that the lowest mode can be $TE_{10}$ or $TE_{01}$ depending on the values of $a$ and $b$, the dimensions of the guide. It is standard practice to have $a > b$ so that $1/a^2 < 1/b^2$ in eq. (12.28). Thus $TE_{10}$ is the lowest mode because $f_{c_{TE_{10}}} = \dfrac{u'}{2a} < f_{c_{TE_{01}}} = \dfrac{u'}{2b}$. This mode is

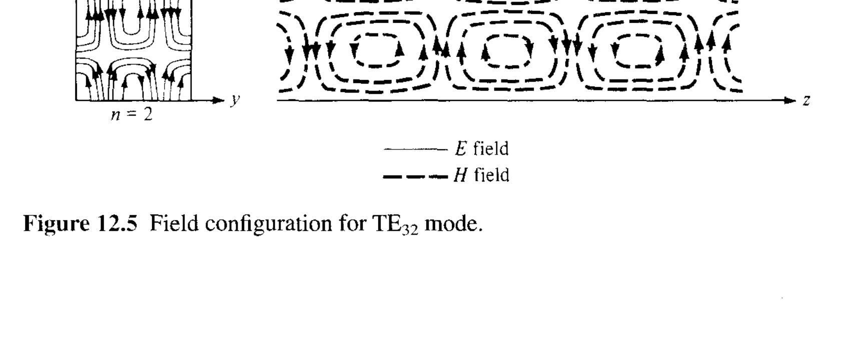


**Figure 12.5** Field configuration for $TE_{32}$ mode.

called the *dominant mode* of the waveguide and is of practical importance. The cutoff frequency for the $TE_{10}$ mode is obtained from eq. (12.28) as ($m = 1, n = 0$)

$$f_{c_{10}} = \frac{u'}{2a} \tag{12.37}$$

and the cutoff wavelength for $TE_{10}$ mode is obtained from eq. (12.29) as

$$\lambda_{c_{10}} = 2a \tag{12.38}$$

Note that from eq. (12.28) the cutoff frequency for $TM_{11}$ is

$$\frac{u'[a^2 + b^2]^{1/2}}{2ab}$$

which is greater than the cutoff frequency for $TE_{10}$. Hence, $TM_{11}$ cannot be regarded as the dominant mode.

> The **dominant mode** is the mode with the lowest cutoff frequency (or longest cutoff wavelength).

Also note that any EM wave with frequency $f < f_{c_{10}}$ (or $\lambda > \lambda_{c_{10}}$) will not be propagated in the guide.

The intrinsic impedance for the TE mode is not the same as for TM modes. From eq. (12.36), it is evident that ($\gamma = j\beta$)

$$\eta_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}} \frac{1}{\sqrt{1 - \left[\frac{f_c}{f}\right]^2}}$$

or

$$\boxed{\eta_{TE} = \frac{\eta'}{\sqrt{1 - \left[\frac{f_c}{f}\right]^2}}} \tag{12.39}$$

Note from eqs. (12.32) and (12.39) that $\eta_{TE}$ and $\eta_{TM}$ are purely resistive and they vary with frequency as shown in Figure 12.6. Also note that

$$\eta_{TE}\, \eta_{TM} = \eta'^2 \tag{12.40}$$

Important equations for TM and TE modes are listed in Table 12.1 for convenience and quick reference.

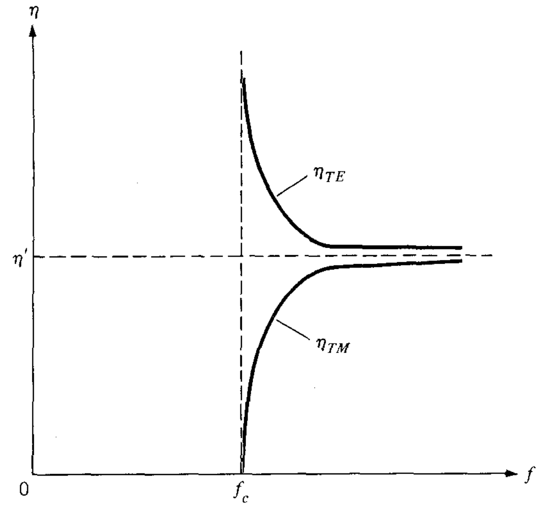


**Figure 12.6** Variation of wave impedance with frequency for TE and TM modes.

**TABLE 12.1** Important Equations for TM and TE Modes

| TM Modes | TE Modes |
|---|---|
| $E_{xs} = -\frac{j\beta}{h^2}\left(\frac{m\pi}{a}\right) E_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ | $E_{xs} = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right) H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ |
| $E_{ys} = -\frac{j\beta}{h^2}\left(\frac{n\pi}{b}\right) E_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ | $E_{ys} = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right) H_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ |
| $E_{zs} = E_o \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ | $E_{zs} = 0$ |
| $H_{xs} = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right) E_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ | $H_{xs} = \frac{j\beta}{h^2}\left(\frac{m\pi}{a}\right) H_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ |
| $H_{ys} = -\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right) E_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ | $H_{ys} = \frac{j\beta}{h^2}\left(\frac{n\pi}{b}\right) H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ |
| $H_{zs} = 0$ | $H_{zs} = H_o \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$ |
| $\eta = \eta'\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$ | $\eta = \frac{\eta'}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$ |

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\lambda_c = \frac{u'}{f_c}$$

$$\beta = \beta'\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$u_p = \frac{\omega}{\beta} = f\lambda$$

where $h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2, u' = \frac{1}{\sqrt{\mu\varepsilon}}, \beta' = \frac{\omega}{u'}, \eta' = \sqrt{\frac{\mu}{\varepsilon}}$

From eqs. (12.22), (12.23), (12.35), and (12.36), we obtain the field patterns for the TM and TE modes. For the dominant $TE_{10}$ mode, $m = 1$ and $n = 0$, so eq. (12.35) becomes

$$H_{zs} = H_o \cos\left(\frac{\pi x}{a}\right) e^{-j\beta z} \tag{12.41}$$

In the time domain,

$$H_z = \text{Re}\,(H_{zs} e^{j\omega t})$$

or

$$H_z = H_o \cos\left(\frac{\pi x}{a}\right) \cos(\omega t - \beta z) \tag{12.42}$$

Similarly, from eq. (12.36),

$$E_y = \frac{\omega\mu a}{\pi} H_o \sin\left(\frac{\pi x}{a}\right) \sin(\omega t - \beta z) \tag{12.43a}$$

$$H_x = -\frac{\beta a}{\pi} H_o \sin\left(\frac{\pi x}{a}\right) \sin(\omega t - \beta z) \tag{12.43b}$$

$$E_z = E_x = H_y = 0 \tag{12.43c}$$

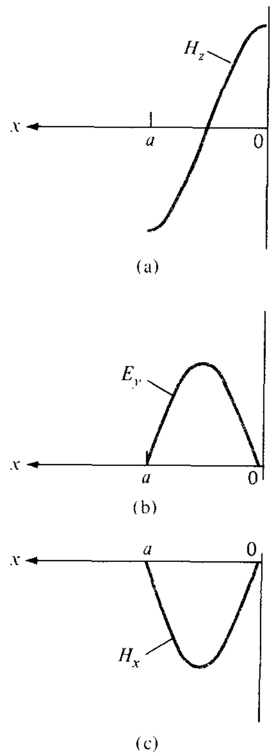


Figure 12.7 Variation of the field components with $x$ for $TE_{10}$ mode.

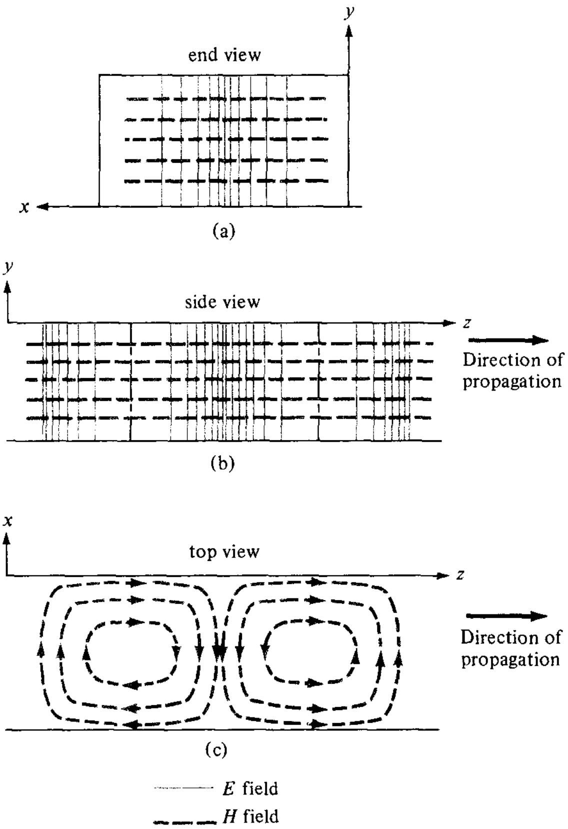


Figure 12.8 Field lines for $TE_{10}$ mode.

The variation of the **E** and **H** fields with $x$ in an $x$–$y$ plane, say plane $\cos(\omega t - \beta z) = 1$ for $H_z$, and plane $\sin(\omega t - \beta z) = 1$ for $E_y$ and $H_x$, is shown in Figure 12.7 for the $TE_{10}$ mode. The corresponding field lines are shown in Figure 12.8.

EXAMPLE 12.1

A rectangular waveguide with dimensions $a = 2.5$ cm, $b = 1$ cm is to operate below 15.1 GHz. How many TE and TM modes can the waveguide transmit if the guide is filled with a medium characterized by $\sigma = 0$, $\varepsilon = 4\,\varepsilon_o$, $\mu_r = 1$? Calculate the cutoff frequencies of the modes.

**Solution:**

The cutoff frequency is given by

$$f_{c_{mn}} = \frac{u'}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

where $a = 2.5$b or $a/b = 2.5$, and

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{2}$$

Hence,

$$f_{c_{mn}} = \frac{c}{4a}\sqrt{m^2 + \frac{a^2}{b^2}n^2}$$
$$= \frac{3 \times 10^8}{4(2.5 \times 10^{-2})}\sqrt{m^2 + 6.25n^2}$$

or

$$f_{c_{mn}} = 3\sqrt{m^2 + 6.25n^2}\ \text{GHz} \tag{12.1.1}$$

We are looking for $f_{c_{mn}} < 15.1$ GHz. A systematic way of doing this is to fix $m$ or $n$ and increase the other until $f_{c_{mn}}$ is greater than 15.1 GHz. From eq. (12.1.1), it is evident that fixing $m$ and increasing $n$ will quickly give us an $f_{c_{mn}}$ that is greater than 15.1 GHz.

For $TE_{01}$ mode ($m = 0, n = 1$), $f_{c_{01}} = 3(2.5) = 7.5$ GHz

$TE_{02}$ mode ($m = 0, n = 2$), $f_{c_{02}} = 3(5) = 15$ GHz

$TE_{03}$ mode, $f_{c_{03}} = 3(7.5) = 22.5$ GHz

Thus for $f_{c_{mn}} < 15.1$ GHz, the maximum $n = 2$. We now fix $n$ and increase $m$ until $f_{c_{mn}}$ is greater than 15.1 GHz.

For $TE_{10}$ mode ($m = 1, n = 0$), $f_{c_{10}} = 3$ GHz

$TE_{20}$ mode, $f_{c_{20}} = 6$ GHz

$TE_{30}$ mode, $f_{c_{30}} = 9$ GHz

$TE_{40}$ mode, $f_{c_{40}} = 12$ GHz

$TE_{50}$ mode, $f_{c_{50}} = 15$ GHz (the same as for $TE_{02}$)

$TE_{60}$ mode, $f_{c_{60}} = 18$ GHz.

that is, for $f_{c_{mn}} < 15.1$ GHz, the maximum $m = 5$. Now that we know the maximum $m$ and $n$, we try other possible combinations in between these maximum values.

For $TE_{11}$, $TM_{11}$ (degenerate modes), $f_{c_{11}} = 3\sqrt{7.25} = 8.078$ GHz

$TE_{21}$, $TM_{21}$, $f_{c_{21}} = 3\sqrt{10.25} = 9.6$ GHz

$TE_{31}$, $TM_{31}$, $f_{c_{31}} = 3\sqrt{15.25} = 11.72$ GHz

$TE_{41}$, $TM_{41}$, $f_{c_{41}} = 3\sqrt{22.25} = 14.14$ GHz

$TE_{12}$, $TM_{12}$, $f_{c_{12}} = 3\sqrt{26} = 15.3$ GHz

Those modes whose cutoff frequencies are less or equal to 15.1 GHz will be transmitted—that is, 11 TE modes and 4 TM modes (all of the above modes except $TE_{12}$, $TM_{12}$, $TE_{60}$, and $TE_{03}$). The cutoff frequencies for the 15 modes are illustrated in the line diagram of Figure 12.9.

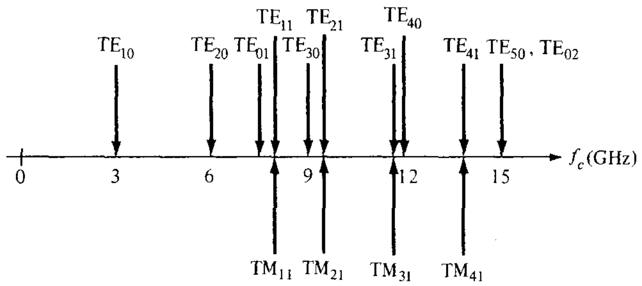


**Figure 12.9** Cutoff frequencies of rectangular waveguide with $a = 2.5b$; for Example 12.1.

**PRACTICE EXERCISE 12.1**

Consider the waveguide of Example 12.1. Calculate the phase constant, phase velocity and wave impedance for $TE_{10}$ and $TM_{11}$ modes at the operating frequency of 15 GHz.

**Answer:** For $TE_{10}$, $\beta = 615.6$ rad/m, $u = 1.531 \times 10^8$ m/s, $\eta_{TE} = 192.4\ \Omega$. For $TM_{11}$, $\beta = 529.4$ rad/m, $u = 1.78 \times 10^8$ m/s, $\eta_{TM} = 158.8\ \Omega$.

**EXAMPLE 12.2**

Write the general instantaneous field expressions for the TM and TE modes. Deduce those for $TE_{01}$ and $TM_{12}$ modes.

**Solution:**

The instantaneous field expressions are obtained from the phasor forms by using

$$\mathbf{E} = \text{Re}\,(\mathbf{E}_s e^{j\omega t}) \qquad \text{and} \qquad \mathbf{H} = \text{Re}\,(\mathbf{H}_s e^{j\omega t})$$

Applying these to eqs. (12.22) and (12.23) while replacing $\gamma$ and $j\beta$ gives the following field components for the TM modes:

$$E_x = \frac{\beta}{h^2}\left[\frac{m\pi}{a}\right] E_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$E_y = \frac{\beta}{h^2}\left[\frac{n\pi}{b}\right] E_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$E_z = E_o \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos(\omega t - z)$$

$$H_x = -\frac{\omega\varepsilon}{h^2}\left[\frac{n\pi}{b}\right] E_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$H_y = \frac{\omega\varepsilon}{h^2}\left[\frac{m\pi}{a}\right] E_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$H_z = 0$$

Similarly, for the TE modes, eqs. (12.35) and (12.36) become

$$E_x = -\frac{\omega\mu}{h^2}\left[\frac{n\pi}{b}\right] H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$E_y = \frac{\omega\mu}{h^2}\left[\frac{m\pi}{a}\right] H_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$E_z = 0$$

$$H_x = -\frac{\beta}{h^2}\left[\frac{m\pi}{a}\right] H_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$H_y = -\frac{\beta}{h^2}\left[\frac{n\pi}{b}\right] H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$H_z = H_o \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos(\omega t - \beta z)$$

For the $TE_{01}$ mode, we set $m = 0, n = 1$ to obtain

$$h^2 = \left[\frac{\pi}{b}\right]^2$$

$$E_x = -\frac{\omega\mu b}{\pi} H_o \sin\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$E_y = 0 = E_z = H_x$$

$$H_y = -\frac{\beta b}{\pi} H_o \sin\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$H_z = H_o \cos\left(\frac{\pi y}{b}\right) \cos(\omega t - \beta z)$$

For the $TM_{12}$ mode, we set $m = 1, n = 2$ to obtain

$$E_x = \frac{\beta}{h^2}\left(\frac{\pi}{a}\right) E_o \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$E_y = \frac{\beta}{h^2}\left(\frac{2\pi}{b}\right) E_o \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$E_z = E_o \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right) \cos(\omega t - \beta z)$$

$$H_x = -\frac{\omega\varepsilon}{h^2}\left(\frac{2\pi}{b}\right) E_o \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi y}{b}\right)\sin(\omega t - \beta z)$$

$$H_y = \frac{\omega\varepsilon}{h^2}\left(\frac{\pi}{a}\right) E_o \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right)\sin(\omega t - \beta z)$$

$$H_z = 0$$

where

$$h^2 = \left[\frac{\pi}{a}\right]^2 + \left[\frac{2\pi}{b}\right]^2$$

**PRACTICE EXERCISE 12.2**

An air-filled 5- by 2-cm waveguide has

$$E_{zs} = 20 \sin 40\pi x \sin 50\pi y \, e^{-j\beta z} \text{ V/m}$$

at 15 GHz.

(a) What mode is being propagated?

(b) Find $\beta$.

(c) Determine $E_y/E_x$.

**Answer:** (a) $TM_{21}$, (b) 241.3 rad/m, (c) $1.25 \tan 40\pi x \cot 50\pi y$.

**EXAMPLE 12.3**

In a rectangular waveguide for which $a = 1.5$ cm, $b = 0.8$ cm, $\sigma = 0$, $\mu = \mu_o$, and $\varepsilon = 4\varepsilon_o$,

$$H_x = 2 \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{3\pi y}{b}\right)\sin(\pi \times 10^{11} t - \beta z) \text{ A/m}$$

Determine

(a) The mode of operation

(b) The cutoff frequency

(c) The phase constant $\beta$

(d) The propagation constant $\gamma$

(e) The intrinsic wave impedance $\eta$.

**Solution:**

(a) It is evident from the given expression for $H_x$ and the field expressions of the last example that $m = 1$, $n = 3$; that is, the guide is operating at $TM_{13}$ or $TE_{13}$. Suppose we choose $TM_{13}$ mode (the possibility of having $TE_{13}$ mode is left as an exercise in Practice Exercise 12.3).

(b)
$$f_{c_{mn}} = \frac{u'}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{2}$$

Hence

$$f_{c_{13}} = \frac{c}{4}\sqrt{\frac{1}{[1.5 \times 10^{-2}]^2} + \frac{9}{[0.8 \times 10^{-2}]^2}}$$
$$= \frac{3 \times 10^8}{4}(\sqrt{0.444 + 14.06}) \times 10^2 = 28.57 \text{ GHz}$$

(c)
$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left[\frac{f_c}{f}\right]^2} = \frac{\omega\sqrt{\varepsilon_r}}{c}\sqrt{1 - \left[\frac{f_c}{f}\right]^2}$$

$$\omega = 2\pi f = \pi \times 10^{11} \quad \text{or} \quad f = \frac{100}{2} = 50 \text{ GHz}$$

$$\beta = \frac{\pi \times 10^{11}(2)}{3 \times 10^8}\sqrt{1 - \left[\frac{28.57}{50}\right]^2} = 1718.81 \text{ rad/m}$$

(d) $\gamma = j\beta = j1718.81$ /m

(e)
$$\eta_{TM_{13}} = \eta'\sqrt{1 - \left[\frac{f_c}{f}\right]^2} = \frac{377}{\sqrt{\varepsilon_r}}\sqrt{1 - \left[\frac{28.57}{50}\right]^2}$$
$$= 154.7\ \Omega$$

**PRACTICE EXERCISE 12.3**

Repeat Example 12.3 if $TE_{13}$ mode is assumed. Determine other field components for this mode.

**Answer:** $f_c = 28.57$ GHz, $\beta = 1718.81$ rad/m, $\mu = j\beta$, $\eta_{TE_{13}} = 229.69\ \Omega$

$$E_x = 2584.1 \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right) \sin(\omega t - \beta z) \text{ V/m}$$

$$E_y = -459.4 \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{3\pi y}{b}\right) \sin(\omega t - \beta z) \text{ V/m}, \qquad E_z = 0$$

$$H_y = 11.25 \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right) \sin(\omega t - \beta z) \text{ A/m}$$

$$H_z = -7.96 \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{3\pi y}{b}\right) \cos(\omega t - \beta z) \text{ A/m}$$

## 12.5 WAVE PROPAGATION IN THE GUIDE

Examination of eq. (12.23) or (12.36) shows that the field components all involve the terms sine or cosine of $(m\pi/a)x$ or $(n\pi/b)y$ times $e^{-\gamma z}$. Since

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \tag{12.44a}$$

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \tag{12.44b}$$

a wave within the waveguide can be resolved into a combination of plane waves reflected from the waveguide walls. For the $TE_{10}$ mode, for example,

$$\begin{aligned} E_{ys} &= -\frac{j\omega\mu a}{\pi}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z} \\ &= -\frac{\omega\mu a}{2\pi}(e^{j\pi x/a} - e^{-j\pi x/a})\,e^{-j\beta z} \\ &= \frac{\omega\mu a}{2\pi}[e^{-j\beta(z+\pi x/\beta a)} - e^{-j\beta(z-\pi x/\beta a)}] \end{aligned} \tag{12.45}$$

The first term of eq. (12.45) represents a wave traveling in the positive $z$-direction at an angle

$$\theta = \tan^{-1}\left(\frac{\pi}{\beta a}\right) \tag{12.46}$$

with the $z$-axis. The second term of eq. (12.45) represents a wave traveling in the positive $z$-direction at an angle $-\theta$. The field may be depicted as a sum of two plane TEM waves propagating along zigzag paths between the guide walls at $x = 0$ and $x = a$ as illustrated in Figure 12.10(a). The decomposition of the $TE_{10}$ mode into two plane waves can be extended to any TE and TM mode. When $n$ and $m$ are both different from zero, four plane waves result from the decomposition.

The wave component in the $z$-direction has a different wavelength from that of the plane waves. This wavelength along the axis of the guide is called the *waveguide wavelength* and is given by (see Problem 12.13)

$$\lambda = \frac{\lambda'}{\sqrt{1 - \left[\frac{f_c}{f}\right]^2}} \tag{12.47}$$

where $\lambda' = u'/f$.

As a consequence of the zigzag paths, we have three types of velocity: the *medium velocity* $u'$, the *phase velocity* $u_p$, and the *group velocity* $u_g$. Figure 12.10(b) illustrates the relationship between the three different velocities. The medium velocity $u' = 1/\sqrt{\mu\varepsilon}$ is as

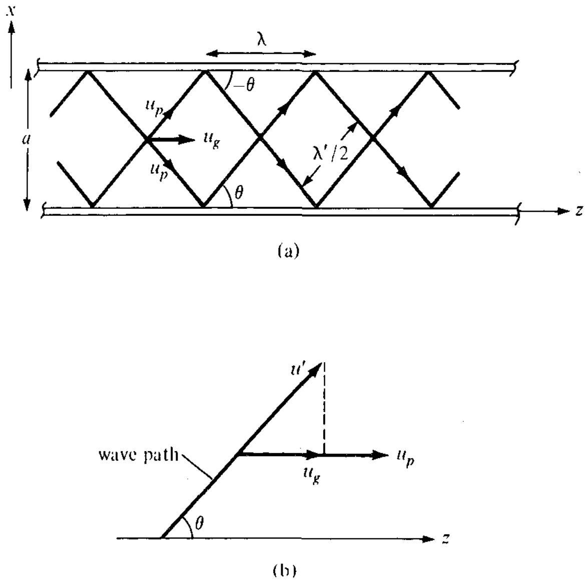


**Figure 12.10** **(a)** Decomposition of $TE_{10}$ mode into two plane waves; **(b)** relationship between $u'$, $u_p$, and $u_g$.

explained in the previous sections. The phase velocity $u_p$ is the velocity at which loci of constant phase are propagated down the guide and is given by eq. (12.31), that is,

$$u_p = \frac{\omega}{\beta} \tag{12.48a}$$

or

$$u_p = \frac{u'}{\cos\theta} = \frac{u'}{\sqrt{1 - \left[\frac{f_c}{f}\right]^2}} \tag{12.48b}$$

This shows that $u_p \geq u'$ since $\cos\theta \leq 1$. If $u' = c$, then $u_p$ is greater than the speed of light in vacuum. Does this violate Einstein's relativity theory that messages cannot travel faster than the speed of light? Not really, because information (or energy) in a waveguide generally does not travel at the phase velocity. Information travels at the group velocity, which must be less than the speed of light. The group velocity $u_g$ is the velocity with which the resultant repeated reflected waves are traveling down the guide and is given by

$$\boxed{u_g = \frac{1}{\partial\beta/\partial\omega}} \tag{12.49a}$$

or

$$u_g = u'\cos\theta = u'\sqrt{1 - \left[\frac{f_c}{f}\right]^2} \tag{12.49b}$$

Although the concept of group velocity is fairly complex and is beyond the scope of this chapter, a group velocity is essentially the velocity of propagation of the wave-packet envelope of a group of frequencies. It is the energy propagation velocity in the guide and is always less than or equal to $u'$. From eqs. (12.48) and (12.49), it is evident that

$$u_p u_g = u'^2 \tag{12.50}$$

This relation is similar to eq. (12.40). Hence the variation of $u_p$ and $u_g$ with frequency is similar to that in Figure 12.6 for $\eta_{TE}$ and $\eta_{TM}$.

**EXAMPLE 12.4**

A standard air-filled rectangular waveguide with dimensions $a = 8.636$ cm, $b = 4.318$ cm is fed by a 4-GHz carrier from a coaxial cable. Determine if a $TE_{10}$ mode will be propagated. If so, calculate the phase velocity and the group velocity.

**Solution:**

For the $TE_{10}$ mode, $f_c = u'/2a$. Since the waveguide is air-filled, $u' = c = 3 \times 10^8$. Hence,

$$f_c = \frac{3 \times 10^8}{2 \times 8.636 \times 10^{-2}} = 1.737 \text{ GHz}$$

As $f = 4$ GHz $> f_c$, the $TE_{10}$ mode will propagate.

$$u_p = \frac{u'}{\sqrt{1 - (f_c/f)^2}} = \frac{3 \times 10^8}{\sqrt{1 - (1.737/4)^2}}$$
$$= 3.33 \times 10^8 \text{ m/s}$$

$$u_g = \frac{u'^2}{u_p} = \frac{9 \times 10^{16}}{3.33 \times 10^8} = 2.702 \times 10^8 \text{ m/s}$$

**PRACTICE EXERCISE 12.4**

Repeat Example 12.4 for the $TM_{11}$ mode.

**Answer:** $12.5 \times 10^8$ m/s, $7.203 \times 10^7$ m/s.

## 12.6 POWER TRANSMISSION AND ATTENUATION

To determine power flow in the waveguide, we first find the average Poynting vector [from eq. (10.68)],

$$\mathcal{P}_{ave} = \frac{1}{2} \text{Re} (\mathbf{E}_s \times \mathbf{H}_s^*) \tag{12.51}$$

In this case, the Poynting vector is along the $z$-direction so that

$$\begin{aligned}\mathcal{P}_{\text{ave}} &= \frac{1}{2}\,\text{Re}\,(E_{xs}H_{ys}^* - E_{ys}H_{xs}^*)\mathbf{a}_z \\ &= \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta}\,\mathbf{a}_z\end{aligned} \tag{12.52}$$

where $\eta = \eta_{\text{TE}}$ for TE modes or $\eta = \eta_{\text{TM}}$ for TM modes. The total average power transmitted across the cross section of the waveguide is

$$\begin{aligned}P_{\text{ave}} &= \int \mathcal{P}_{\text{ave}} \cdot d\mathbf{S} \\ &= \int_{x=0}^{a}\int_{y=0}^{b} \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta}\,dy\,dx\end{aligned} \tag{12.53}$$

Of practical importance is the attenuation in a lossy waveguide. In our analysis thus far, we have assumed lossless waveguides ($\sigma = 0$, $\sigma_c \simeq \infty$) for which $\alpha = 0$, $\gamma = j\beta$. When the dielectric medium is lossy ($\sigma \neq 0$) and the guide walls are not perfectly conducting ($\sigma_c \neq \infty$), there is a continuous loss of power as a wave propagates along the guide. According to eqs. (10.69) and (10.70), the power flow in the guide is of the form

$$P_{\text{ave}} = P_o e^{-2\alpha z} \tag{12.54}$$

In order that energy be conserved, the rate of decrease in $P_{\text{ave}}$ must equal the time average power loss $P_L$ per unit length, that is,

$$P_L = -\frac{dP_{\text{ave}}}{dz} = 2\alpha P_{\text{ave}}$$

or

$$\alpha = \frac{P_L}{2P_{\text{ave}}} \tag{12.55}$$

In general,

$$\alpha = \alpha_c + \alpha_d \tag{12.56}$$

where $\alpha_c$ and $\alpha_d$ are attenuation constants due to ohmic or conduction losses ($\sigma_c \neq \infty$) and dielectric losses ($\sigma \neq 0$), respectively.

To determine $\alpha_d$, recall that we started with eq. (12.1) assuming a lossless dielectric medium ($\sigma = 0$). For a lossy dielectric, we need to incorporate the fact that $\sigma \neq 0$. All our equations still hold except that $\gamma = j\beta$ needs to be modified. This is achieved by replacing $\varepsilon$ in eq. (12.25) by the complex permittivity of eq. (10.40). Thus, we obtain

$$\gamma = \alpha_d + j\beta_d = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon_c} \tag{12.57}$$

where

$$\varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\frac{\sigma}{\omega} \tag{12.58}$$

Substituting eq. (12.58) into eq. (12.57) and squaring both sides of the equation, we obtain

$$\gamma^2 = \alpha_d^2 - \beta_d^2 + 2j\alpha_d\beta_d = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon + j\omega\mu\sigma$$

Equating real and imaginary parts,

$$\alpha_d^2 - \beta_d^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon \tag{12.59a}$$

$$2\alpha_d\beta_d = \omega\mu\sigma \qquad \text{or} \qquad \alpha_d = \frac{\omega\mu\sigma}{2\beta_d} \tag{12.59b}$$

Assuming that $\alpha_d^2 \ll \beta_d^2$, $\alpha_d^2 - \beta_d^2 \simeq -\beta_d^2$, so eq. (12.59a) gives

$$\begin{aligned}\beta_d &= \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}\\ &= \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}\end{aligned} \tag{12.60}$$

which is the same as $\beta$ in eq. (12.30). Substituting eq. (12.60) into eq. (12.59b) gives

$$\boxed{\alpha_d = \frac{\sigma\eta'}{2\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}} \tag{12.61}$$

where $\eta' = \sqrt{\mu/\varepsilon}$.

The determination of $\alpha_c$ for $TM_{mn}$ and $TE_{mn}$ modes is time consuming and tedious. We shall illustrate the procedure by finding $\alpha_c$ for the $TE_{10}$ mode. For this mode, only $E_y$, $H_x$, and $H_z$ exist. Substituting eq. (12.43a) into eq. (12.53) yields

$$\begin{aligned}P_{\text{ave}} &= \int_{x=0}^{a}\int_{y=0}^{b}\frac{|E_{ys}|^2}{2\eta}\,dx\,dy = \frac{\omega^2\mu^2a^2H_o^2}{2\pi^2\eta}\int_0^b dy\int_0^a \sin^2\frac{\pi x}{a}\,dx\\ P_{\text{ave}} &= \frac{\omega^2\mu^2a^3H_o^2 b}{4\pi^2\eta}\end{aligned} \tag{12.62}$$

The total power loss per unit length in the walls is

$$\begin{aligned}P_L &= P_L\big|_{y=0} + P_L\big|_{y=b} + P_L\big|_{x=0} + P_L\big|_{x=a}\\ &= 2(P_L\big|_{y=0} + P_L\big|_{x=0})\end{aligned} \tag{12.63}$$

since the same amount is dissipated in the walls $y = 0$ and $y = b$ or $x = 0$ and $x = a$. For the wall $y = 0$,

$$
\begin{aligned}
P_L \big|_{y=0} &= \frac{1}{2} \mathrm{Re} \left[ \eta_c \int (|H_{xs}|^2 + |H_{zs}|^2)\, dx \right] \Bigg|_{y=0} \\
&= \frac{1}{2} R_s \left[ \int_0^a \frac{\beta^2 a^2}{\pi^2} H_o^2 \sin^2 \frac{\pi x}{a}\, dx + \int_0^a H_o^2 \cos^2 \frac{\pi x}{a}\, dx \right] \\
&= \frac{R_s a H_o^2}{4} \left( 1 + \frac{\beta^2 a^2}{\pi^2} \right)
\end{aligned}
\tag{12.64}
$$

where $R_s$ is the real part of the intrinsic impedance $\eta_c$ of the conducting wall. From eq. (10.56),

$$
R_s = \frac{1}{\sigma_c \delta} \tag{12.65}
$$

where $\delta$ is the skin depth. $R_s$ is the skin resistance of the wall; it may be regarded as the resistance of 1 m by $\delta$ by 1 m of the conducting material. For the wall $x = 0$,

$$
\begin{aligned}
P_L \big|_{x=0} &= \frac{1}{2} \mathrm{Re} \left[ \eta_c \int (|H_{zs}|^2)\, dy \right] \Big|_{x=0} = \frac{1}{2} R_s \int_0^b H_o^2\, dy \\
&= \frac{R_s b H_o^2}{2}
\end{aligned}
\tag{12.66}
$$

Substituting eqs. (12.64) and (12.66) into eq. (12.63) gives

$$
P_L = R_s H_o^2 \left[ b + \frac{a}{2} \left( 1 + \frac{\beta^2 a^2}{\pi^2} \right) \right] \tag{12.67}
$$

Finally, substituting eqs. (12.62) and (12.67) into eq. (12.55),

$$
\alpha_c = \frac{R_s H_o^2 \left[ b + \frac{a}{2} \left( 1 + \frac{\beta^2 a^2}{\pi^2} \right) \right] 2\pi^2 \eta}{\omega^2 \mu^2 a^3 H_o^2 b} \tag{12.68a}
$$

It is convenient to express $\alpha_c$ in terms of $f$ and $f_c$. After some manipulations, we obtain for the $TE_{10}$ mode

$$
\alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - \left[ \frac{f_c}{f} \right]^2}} \left( \frac{1}{2} + \frac{b}{a} \left[ \frac{f_c}{f} \right]^2 \right) \tag{12.68b}
$$

By following the same procedure, the attenuation constant for the $TE_{mn}$ modes ($n \neq 0$) can be obtained as

$$\alpha_c \big|_{TE} = \frac{2R_s}{b\eta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2}} \left[ \left(1 + \frac{b}{a}\right)\left[\frac{f_c}{f}\right]^2 + \frac{\frac{b}{a}\left(\frac{b}{a}m^2 + n^2\right)}{\frac{b^2}{a^2}m^2 + n^2} \left(1 - \left[\frac{f_c}{f}\right]^2\right) \right] \tag{12.69}$$

and for the $TM_{mn}$ modes as

$$\alpha_c \big|_{TM} = \frac{2R_s}{b\eta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2}} \frac{(b/a)^3 m^2 + n^2}{(b/a)^2 m^2 + n^2} \tag{12.70}$$

The total attenuation constant $\alpha$ is obtained by substituting eqs. (12.61) and (12.69) or (12.70) into eq. (12.56).

## 12.7 WAVEGUIDE CURRENT AND MODE EXCITATION

For either TM or TE modes, the surface current density **K** on the walls of the waveguide may be found using

$$\mathbf{K} = \mathbf{a}_n \times \mathbf{H} \tag{12.71}$$

where $\mathbf{a}_n$ is the unit outward normal to the wall and **H** is the field intensity evaluated on the wall. The current flow on the guide walls for $TE_{10}$ mode propagation can be found using eq. (12.71) with eqs. (12.42) and (12.43). The result is sketched in Figure 12.11.

The surface charge density $\rho_S$ on the walls is given by

$$\rho_S = \mathbf{a}_n \cdot \mathbf{D} = \mathbf{a}_n \cdot \varepsilon\mathbf{E} \tag{12.72}$$

where **E** is the electric field intensity evaluated on the guide wall.

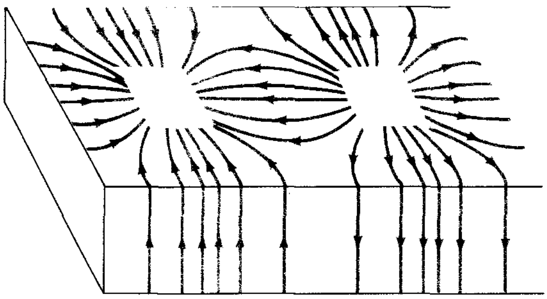

Figure 12.11 Surface current on guide walls for $TE_{10}$ mode.

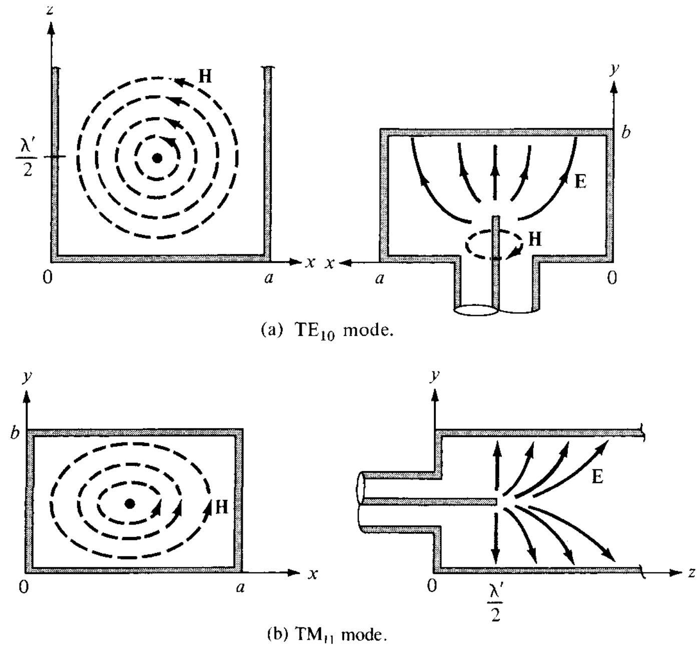


**Figure 12.12** Excitation of modes in a rectangular waveguide.

A waveguide is usually fed or excited by a coaxial line or another waveguide. Most often, a probe (central conductor of a coaxial line) is used to establish the field intensities of the desired mode and achieve a maximum power transfer. The probe is located so as to produce **E** and **H** fields that are roughly parallel to the lines of **E** and **H** fields of the desired mode. To excite the $TE_{10}$ mode, for example, we know from eq. (12.43a) that $E_y$ has maximum value at $x = a/2$. Hence, the probe is located at $x = a/2$ to excite the $TE_{10}$ mode as shown in Figure 12.12(a), where the field lines are similar to those of Figure 12.8. Similarly, the $TM_{11}$ mode is launched by placing the probe along the $z$-direction as in Figure 12.12(b).

**EXAMPLE 12.5**

An air-filled rectangular waveguide of dimensions $a = 4$ cm, $b = 2$ cm transports energy in the dominant mode at a rate of 2 mW. If the frequency of operation is 10 GHz, determine the peak value of the electric field in the waveguide.

**Solution:**

The dominant mode for $a > b$ is $TE_{10}$ mode. The field expressions corresponding to this mode ($m = 1, n = 0$) are in eq. (12.36) or (12.43), namely

$$E_{xs} = 0, \qquad E_{ys} = -jE_o \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z}, \qquad \text{where } E_o = \frac{\omega\mu a}{\pi} H_o$$

$$f_c = \frac{u'}{2a} = \frac{3 \times 10^8}{2(4 \times 10^{-2})} = 3.75 \text{ GHz}$$

$$\eta = \eta_{TE} = \frac{\eta'}{\sqrt{1 - \left[\frac{f_c}{f}\right]^2}} = \frac{377}{\sqrt{1 - \left[\frac{3.75}{10}\right]^2}} = 406.7 \ \Omega$$

From eq. (12.53), the average power transmitted is

$$P_{\text{ave}} = \int_{y=0}^{b} \int_{x=0}^{a} \frac{|E_{ys}|^2}{2\eta} \, dx \, dy = \frac{E_o^2}{2\eta} \int_0^b dy \int_0^a \sin^2\left(\frac{\pi x}{a}\right) dz$$
$$= \frac{E_o^2 \, ab}{4\eta}$$

Hence,

$$E_o^2 = \frac{4\eta P_{\text{ave}}}{ab} = \frac{4(406.7) \times 2 \times 10^{-3}}{8 \times 10^{-4}} = 4067$$

$$E_o = 63.77 \text{ V/m}$$

**PRACTICE EXERCISE 12.5**

In Example 12.5, calculate the peak value $H_o$ of the magnetic field in the guide if $a = 2$ cm, $b = 4$ cm while other things remain the same.

**Answer:** 63.34 mA/m.

**EXAMPLE 12.6**

A copper-plated waveguide ($\sigma_c = 5.8 \times 10^7$ S/m) operating at 4.8 GHz is supposed to deliver a minimum power of 1.2 kW to an antenna. If the guide is filled with polystyrene ($\sigma = 10^{-17}$ S/m, $\varepsilon = 2.55\varepsilon_o$) and its dimensions are $a = 4.2$ cm, $b = 2.6$ cm, calculate the power dissipated in a length 60 cm of the guide in the $TE_{10}$ mode.

**Solution:**

Let

$P_d$ = power loss or dissipated
$P_a$ = power delivered to the antenna
$P_o$ = input power to the guide

so that $P_o = P_d + P_a$
From eq. (12.54),

$$P_a = P_o e^{-2\alpha z}$$

Hence,

$$P_a = (P_d + P_a)\, e^{-2\alpha z}$$

or

$$P_d = P_a(e^{2\alpha z} - 1)$$

Now we need to determine $\alpha$ from

$$\alpha = \alpha_d + \alpha_c$$

From eq. (12.61),

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1 - \left[\frac{f_c}{f}\right]^2}}$$

Since the loss tangent

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-17}}{2\pi \times 4.8 \times 10^9 \times \frac{10^{-9}}{36\pi} \times 2.55}$$
$$= 1.47 \times 10^{-17} \ll 1 \qquad \text{(lossless dielectric medium)}$$

then

$$\eta' \simeq \sqrt{\frac{\mu}{\varepsilon}} = \frac{377}{\sqrt{\varepsilon_r}} = 236.1$$

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = 1.879 \times 10^8 \text{ m/s}$$

$$f_c = \frac{u'}{2a} = \frac{1.879 \times 10^8}{2 \times 4.2 \times 10^{-2}} = 2.234 \text{ GHz}$$

$$\alpha_d = \frac{10^{-17} \times 236.1}{2\sqrt{1 - \left[\frac{2.234}{4.8}\right]^2}}$$
$$\alpha_d = 1.334 \times 10^{-15} \text{ Np/m}$$

For the $TE_{10}$ mode, eq. (12.68b) gives

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1 - \left[\frac{f_c}{f}\right]^2}}\left(0.5 + \frac{b}{a}\left[\frac{f_c}{f}\right]^2\right)$$

where

$$R_s = \frac{1}{\sigma_c \delta} = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi \times 4.8 \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}}$$
$$= 1.808 \times 10^{-2}\ \Omega$$

Hence

$$\alpha_c = \frac{2 \times 1.808 \times 10^{-2}\left(0.5 + \frac{2.6}{4.2}\left[\frac{2.234}{4.8}\right]^2\right)}{2.6 \times 10^{-2} \times 236.1\sqrt{1 - \left[\frac{2.234}{4.8}\right]^2}}$$
$$= 4.218 \times 10^{-3}\ \text{Np/m}$$

Note that $\alpha_d \ll \alpha_c$, showing that the loss due to the finite conductivity of the guide walls is more important than the loss due to the dielectric medium. Thus

$$\alpha = \alpha_d + \alpha_c \simeq \alpha_c = 4.218 \times 10^{-3}\ \text{Np/m}$$

and the power dissipated is

$$P_d = P_a\,(e^{2\alpha z} - 1) = 1.2 \times 10^3(e^{2\times4.218\times10^{-3}\times0.6} - 1)$$
$$= 6.089\ \text{W}$$

**PRACTICE EXERCISE 12.6**

A brass waveguide ($\sigma_c = 1.1 \times 10^7$ mhos/m) of dimensions $a = 4.2$ cm, $b = 1.5$ cm is filled with Teflon ($\varepsilon_r = 2.6$, $\sigma = 10^{-15}$ mhos/m). The operating frequency is 9 GHz. For the $TE_{10}$ mode:

(a) Calculate $\alpha_d$ and $\alpha_c$.

(b) What is the loss in decibels in the guide if it is 40 cm long?

**Answer:** (a) $1.206 \times 10^{-13}$ Np/m, $1.744 \times 10^{-2}$ Np/m, (b) 0.0606 dB.

EXAMPLE 12.7

Sketch the field lines for the $TM_{11}$ mode. Derive the instantaneous expressions for the surface current density of this mode.

**Solution:**

From Example 12.2, we obtain the fields for $TM_{11}$ mode ($m = 1, n = 1$) as

$$E_x = \frac{\beta}{h^2}\left(\frac{\pi}{a}\right) E_o \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$E_y = \frac{\beta}{h^2}\left(\frac{\pi}{b}\right) E_o \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$E_z = E_o \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \cos(\omega t - \beta z)$$

$$H_x = -\frac{\omega\varepsilon}{h^2}\left(\frac{\pi}{b}\right) E_o \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$H_y = \frac{\omega\varepsilon}{h^2}\left(\frac{\pi}{a}\right) E_o \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)$$

$$H_z = 0$$

For the electric field lines,

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{a}{b} \tan\left(\frac{\pi x}{a}\right) \cot\left(\frac{\pi y}{b}\right)$$

For the magnetic field lines,

$$\frac{dy}{dx} = \frac{H_y}{H_x} = -\frac{b}{a} \cot\left(\frac{\pi x}{a}\right) \tan\left(\frac{\pi y}{b}\right)$$

Notice that $(E_y/E_x)(H_y/H_x) = -1$, showing that electric and magnetic field lines are mutually orthogonal. This should also be observed in Figure 12.13 where the field lines are sketched.

The surface current density on the walls of the waveguide is given by

$$\mathbf{K} = \mathbf{a}_n \times \mathbf{H} = \mathbf{a}_n \times (H_x, H_y, 0)$$

At $x = 0$, $\mathbf{a}_n = \mathbf{a}_x$, $\mathbf{K} = H_y(0, y, z, t)\, \mathbf{a}_z$, that is,

$$\mathbf{K} = \frac{\omega\varepsilon}{h^2}\left(\frac{\pi}{a}\right) E_o \sin\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)\, \mathbf{a}_z$$

At $x = a$, $\mathbf{a}_n = -\mathbf{a}_x$, $\mathbf{K} = -H_y(a, y, z, t)\, \mathbf{a}_z$
or

$$\mathbf{K} = \frac{\omega\varepsilon}{h^2}\left(\frac{\pi}{a}\right) E_o \sin\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)\, \mathbf{a}_z$$

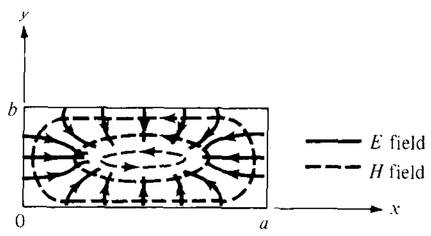


**Figure 12.13** Field lines for $TM_{11}$ mode; for Example 12.7.

At $y = 0$, $\mathbf{a}_n = \mathbf{a}_y$, $\mathbf{K} = -H_x(x, 0, z, t)\,\mathbf{a}_z$
or

$$\mathbf{K} = \frac{\omega\varepsilon}{h^2}\left(\frac{\pi}{b}\right) E_o \sin\left(\frac{\pi x}{a}\right) \sin(\omega t - \beta z)\,\mathbf{a}_z$$

At $y = b$, $\mathbf{a}_n = -\mathbf{a}_y$, $\mathbf{K} = H_x(x, b, z, t)\,\mathbf{a}_z$
or

$$\mathbf{K} = \frac{\omega\varepsilon}{h^2}\left(\frac{\pi}{b}\right) E_o \sin\left(\frac{\pi x}{a}\right) \sin(\omega t - \beta z)\,\mathbf{a}_z$$

**PRACTICE EXERCISE 12.7**

Sketch the field lines for the $TE_{11}$ mode.

**Answer:** See Figure 12.14. The strength of the field at any point is indicated by the density of the lines; the field is strongest (or weakest) where the lines are closest together (or farthest apart).

## 12.8 WAVEGUIDE RESONATORS

Resonators are primarily used for energy storage. At high frequencies (100 MHz and above) the *RLC* circuit elements are inefficient when used as resonators because the dimensions of the circuits are comparable with the operating wavelength, and consequently, unwanted radiation takes place. Therefore, at high frequencies the *RLC* resonant circuits

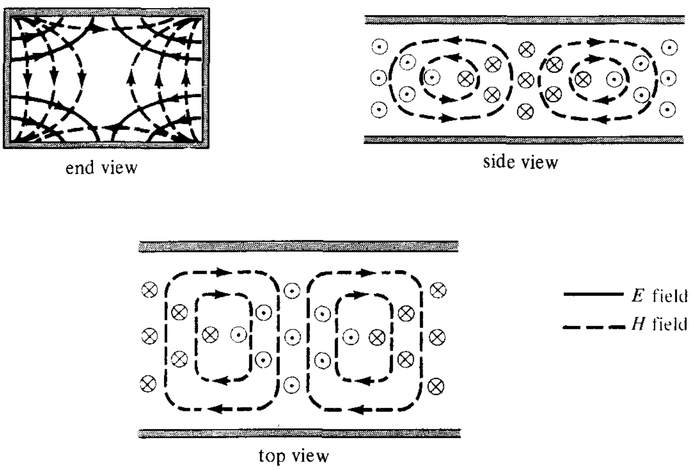


**Figure 12.14** For Practice Exercise 12.7; for $TE_{11}$ mode.

are replaced by electromagnetic cavity resonators. Such resonator cavities are used in klystron tubes, bandpass filters, and wave meters. The microwave oven essentially consists of a power supply, a waveguide feed, and an oven cavity.

Consider the rectangular cavity (or closed conducting box) shown in Figure 12.15. We notice that the cavity is simply a rectangular waveguide shorted at both ends. We therefore expect to have standing wave and also TM and TE modes of wave propagation. Depending on how the cavity is excited, the wave can propagate in the $x$-, $y$-, or $z$-direction. We will choose the $+z$-direction as the "direction of wave propagation." In fact, there is no wave propagation. Rather, there are standing waves. We recall from Section 10.8 that a standing wave is a combination of two waves traveling in opposite directions.

## A. TM Mode to $z$

For this case, $H_z = 0$ and we let

$$E_{zs}(x, y, z) = X(x)\, Y(y)\, Z(z) \tag{12.73}$$

be the production solution of eq. (12.1). We follow the same procedure taken in Section 12.2 and obtain

$$X(x) = c_1 \cos k_x x + c_2 \sin k_x x \tag{12.74a}$$

$$Y(y) = c_3 \cos k_y y + c_4 \sin k_y y \tag{12.74b}$$

$$Z(z) = c_5 \cos k_z z + c_6 \sin k_z z \tag{12.74c}$$

where

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon \tag{12.75}$$

The boundary conditions are:

$$E_z = 0 \quad \text{at} \quad x = 0, a \tag{12.76a}$$

$$E_z = 0 \quad \text{at} \quad y = 0, b \tag{12.76b}$$

$$E_y = 0, E_x = 0 \quad \text{at} \quad z = 0, c \tag{12.76c}$$

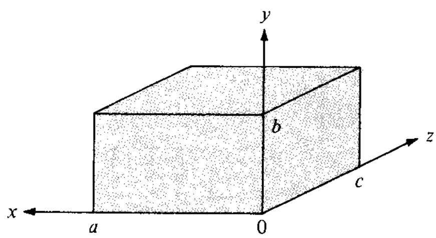


Figure 12.15 Rectangular cavity.

As shown in Section 12.3, the conditions in eqs. (12.7a, b) are satisfied when $c_1 = 0 = c_3$ and

$$k_x = \frac{m\pi}{a}, \qquad k_y = \frac{n\pi}{b} \tag{12.77}$$

where $m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$. To invoke the conditions in eq. (12.76c), we notice that eq. (12.14) (with $H_{zs} = 0$) yields

$$j\omega\varepsilon E_{xs} = \frac{1}{j\omega\mu}\left(\frac{\partial^2 E_{xs}}{\partial z^2} - \frac{\partial^2 E_{zs}}{\partial z\, \partial x}\right) \tag{12.78}$$

Similarly, combining eqs. (12.13a) and (12.13d) (with $H_{zs} = 0$) results in

$$j\omega\varepsilon E_{ys} = \frac{1}{-j\omega\mu}\left(\frac{\partial^2 E_{zs}}{\partial y\, \partial z} - \frac{\partial^2 E_{ys}}{\partial z^2}\right) \tag{12.79}$$

From eqs. (12.78) and (12.79), it is evident that eq. (12.76c) is satisfied if

$$\frac{\partial E_{zs}}{\partial z} = 0 \qquad \text{at} \qquad z = 0, c \tag{12.80}$$

This implies that $c_6 = 0$ and $\sin k_z c = 0 = \sin p\pi$. Hence,

$$k_z = \frac{p\pi}{c} \tag{12.81}$$

where $p = 0, 1, 2, 3, \ldots$. Substituting eqs. (12.77) and (12.81) into eq. (12.74) yields

$$\boxed{E_{zs} = E_o \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)} \tag{12.82}$$

where $E_o = c_2 c_4 c_5$. Other field components are obtained from eqs. (12.82) and (12.13). The phase constant $\beta$ is obtained from eqs. (12.75), (12.77), and (12.81) as

$$\beta^2 = k^2 = \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2 + \left[\frac{p\pi}{c}\right]^2 \tag{12.83}$$

Since $\beta^2 = \omega^2\mu\varepsilon$, from eq. (12.83), we obtain the *resonant frequency* $f_r$

$$2\pi f_r = \omega_r = \frac{\beta}{\sqrt{\mu\varepsilon}} = \beta u'$$

or

$$\boxed{f_r = \frac{u'}{2}\sqrt{\left[\frac{m}{a}\right]^2 + \left[\frac{n}{b}\right]^2 + \left[\frac{p}{c}\right]^2}} \tag{12.84}$$

The corresponding resonant wavelength is

$$\lambda_r = \frac{u'}{f_r} = \frac{2}{\sqrt{\left[\frac{m}{a}\right]^2 + \left[\frac{n}{b}\right]^2 + \left[\frac{p}{c}\right]^2}} \tag{12.85}$$

From eq. (12.84), we notice that the lowest-order TM mode is $TM_{110}$.

## B. TE Mode to $z$

In this case, $E_z = 0$ and

$$H_{zs} = (b_1 \cos k_x x + b_2 \sin k_x x)(b_3 \cos k_y y + b_4 \sin k_y y)$$
$$(b_5 \cos k_z z + \sin k_z z) \tag{12.86}$$

The boundary conditions in eq. (12.76c) combined with eq. (12.13) yields

$$H_{zs} = 0 \quad \text{at} \quad z = 0, c \tag{12.87a}$$

$$\frac{\partial H_{zs}}{\partial x} = 0 \quad \text{at} \quad x = 0, a \tag{12.87b}$$

$$\frac{\partial H_{zs}}{\partial y} = 0 \quad \text{at} \quad y = 0, b \tag{12.87c}$$

Imposing the conditions in eq. (12.87) on eq. (12.86) in the same manner as for TM mode to $z$ leads to

$$H_{zs} = H_o \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{c}\right) \tag{12.88}$$

where $m = 0, 1, 2, 3, \ldots, n = 0, 1, 2, 3, \ldots$, and $p = 1, 2, 3, \ldots$. Other field components can be obtained from eqs. (12.13) and (12.88). The resonant frequency is the same as that of eq. (12.84) except that $m$ or $n$ (but not both at the same time) can be zero for TE modes. The reason why $m$ and $n$ cannot be zero at the same time is that the field components will be zero if they are zero. The mode that has the lowest resonant frequency for a given cavity size *(a, b, c)* is the *dominant mode.* If $a > b < c$, it implies that $1/a < 1/b > 1/c$ and hence the dominant mode is $TE_{101}$. Note that for $a > b < c$, the resonant frequency of $TM_{110}$ mode is higher than that for $TE_{101}$ mode; hence, $TE_{101}$ is dominant. When different modes have the same resonant frequency, we say that the modes are *degenerate;* one mode will dominate others depending on how the cavity is excited.

A practical resonant cavity has walls with finite conductivity $\sigma_c$ and is, therefore, capable of losing stored energy. The *quality factor Q* is a means of determining the loss.

The **quality factor** is also a measure of the bandwidth of the cavity resonator.

It may be defined as

$$Q = 2\pi \cdot \frac{\text{Time average energy stored}}{\text{Energy loss per cycle of oscillation}}$$
$$= 2\pi \cdot \frac{W}{P_L T} = \omega \frac{W}{P_L} \tag{12.89}$$

where $T = 1/f$ = the period of oscillation, $P_L$ is the time average power loss in the cavity, and $W$ is the total time average energy stored in electric and magnetic fields in the cavity. $Q$ is usually very high for a cavity resonator compared with that for an *RLC* resonant circuit. By following a procedure similar to that used in deriving $\alpha_c$ in Section 12.6, it can be shown that the quality factor for the dominant $TE_{101}$ is given by[3]

$$\boxed{Q_{TE_{101}} = \frac{(a^2 + c^2)abc}{\delta[2b(a^3 + c^3) + ac(a^2 + c^2)]}} \tag{12.90}$$

where $\delta = \dfrac{1}{\sqrt{\pi f_{101} \mu_o \sigma_c}}$ is the skin depth of the cavity walls.

**EXAMPLE 12.8**

An air-filled resonant cavity with dimensions $a = 5$ cm, $b = 4$ cm, and $c = 10$ cm is made of copper ($\sigma_c = 5.8 \times 10^7$ mhos/m). Find

(a) The five lowest order modes

(b) The quality factor for $TE_{101}$ mode

**Solution:**

(a) The resonant frequency is given by

$$f_r = \frac{u'}{2}\sqrt{\left[\frac{m}{a}\right]^2 + \left[\frac{n}{b}\right]^2 + \left[\frac{p}{c}\right]^2}$$

where

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = c$$

[3]For the proof, see S. V. Marshall and G. G. Skitek, *Electromagnetic Concepts and Applications,* 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1990, pp. 440–442.

Hence

$$f_r = \frac{3 \times 10^8}{2}\sqrt{\left[\frac{m}{5 \times 10^{-2}}\right]^2 + \left[\frac{n}{4 \times 10^{-2}}\right]^2 + \left[\frac{p}{10 \times 10^{-2}}\right]^2}$$
$$= 15\sqrt{0.04m^2 + 0.0625n^2 + 0.01p^2}\ \text{GHz}$$

Since $c > a > b$ or $1/c < 1/a < 1/b$, the lowest order mode is $TE_{101}$. Notice that $TM_{101}$ and $TE_{100}$ do not exist because $m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots,$ and $p = 0, 1, 2, 3, \ldots$ for the TM modes, and $m = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots,$ and $p = 1, 2, 3, \ldots$ for the TE modes. The resonant frequency for the $TE_{101}$ mode is

$$f_{r_{101}} = 15\sqrt{0.04 + 0 + 0.01} = 3.335\ \text{GHz}$$

The next higher mode is $TE_{011}$ ($TM_{011}$ does not exist), with

$$f_{r_{011}} = 15\sqrt{0 + 0.0625 + 0.01} = 4.04\ \text{GHz}$$

The next mode is $TE_{102}$ ($TM_{102}$ does not exist), with

$$f_{r_{102}} = 15\sqrt{0.04 + 0 + 0.04} = 4.243\ \text{GHz}$$

The next mode is $TM_{110}$ ($TE_{110}$ does not exist), with

$$f_{r_{110}} = 15\sqrt{0.04 + 0.0625 + 0} = 4.8\ \text{GHz}$$

The next two modes are $TE_{111}$ and $TM_{111}$ (degenerate modes), with

$$f_{r_{111}} = 15\sqrt{0.04 + 0.0625 + 0.01} = 5.031\ \text{GHz}$$

The next mode is $TM_{103}$ with

$$f_{r_{103}} = 15\sqrt{0.04 + 0 + 0.09} = 5.408\ \text{GHz}$$

Thus the five lowest order modes in ascending order are

| | |
|---|---|
| $TE_{101}$ | (3.35 GHz) |
| $TE_{011}$ | (4.04 GHz) |
| $TE_{102}$ | (4.243 GHz) |
| $TM_{110}$ | (4.8 GHz) |
| $TE_{111}$ or $TM_{111}$ | (5.031 GHz) |

(b) The quality factor for $TE_{101}$ is given by

$$Q_{TE_{101}} = \frac{(a^2 + c^2)\,abc}{\delta[2b(a^3 + c^3) + ac(a^2 + c^2)]}$$
$$= \frac{(25 + 100)\,200 \times 10^{-2}}{\delta[8(125 + 1000) + 50(25 + 100)]}$$
$$= \frac{1}{61\delta} = \frac{\sqrt{\pi f_{101}\,\mu_o \sigma_c}}{61}$$
$$= \frac{\sqrt{\pi(3.35 \times 10^9)\,4\pi \times 10^{-7}\,(5.8 \times 10^7)}}{61}$$
$$= 14{,}358$$

**PRACTICE EXERCISE 12.8**

If the resonant cavity of Example 12.8 is filled with a lossless material ($\mu_r = 1$, $\varepsilon_r = 3$), find the resonant frequency $f_r$ and the quality factor for $TE_{101}$ mode.

**Answer:** 1.936 GHz, $1.093 \times 10^4$

## SUMMARY

1. Waveguides are structures used in guiding EM waves at high frequencies. Assuming a lossless rectangular waveguide ($\sigma_c \simeq \infty$, $\sigma \simeq 0$), we apply Maxwell's equations in analyzing EM wave propagation through the guide. The resulting partial differential equation is solved using the method of separation of variables. On applying the boundary conditions on the walls of the guide, the basic formulas for the guide are obtained for different modes of operation.
2. Two modes of propagation (or field patterns) are the $TM_{mn}$ and $TE_{mn}$ where $m$ and $n$ are positive integers. For TM modes, $m = 1, 2, 3, \ldots$, and $n = 1, 2, 3, \ldots$ and for TE modes, $m = 0, 1, 2, \ldots$, and $n = 0, 1, 2, \ldots$, $n = m \neq 0$.
3. Each mode of propagation has associated propagation constant and cutoff frequency. The propagation constant $\gamma = \alpha + j\beta$ does not only depend on the constitutive parameters ($\varepsilon$, $\mu$, $\sigma$) of the medium as in the case of plane waves in an unbounded space, it also depends on the cross-sectional dimensions ($a$, $b$) of the guide. The cutoff frequency is the frequency at which $\gamma$ changes from being purely real (attenuation) to purely imaginary (propagation). The dominant mode of operation is the lowest mode possible. It is the mode with the lowest cutoff frequency. If $a > b$, the dominant mode is $TE_{10}$.
4. The basic equations for calculating the cutoff frequency $f_c$, phase constant $\beta$, and phase velocity $u$ are summarized in Table 12.1. Formulas for calculating the attenuation constants due to lossy dielectric medium and imperfectly conducting walls are also provided.
5. The group velocity (or velocity of energy flow) $u_g$ is related to the phase velocity $u_p$ of the wave propagation by

$$u_p u_g = u'^2$$

where $u' = 1/\sqrt{\mu\varepsilon}$ is the medium velocity—i.e., the velocity of the wave in the dielectric medium unbounded by the guide. Although $u_p$ is greater than $u'$, $u_p$ does not exceed $u'$.
6. The mode of operation for a given waveguide is dictated by the method of excitation.
7. A waveguide resonant cavity is used for energy storage at high frequencies. It is nothing but a waveguide shorted at both ends. Hence its analysis is similar to that of a waveguide. The resonant frequency for both the TE and TM modes to $z$ is given by

$$f_r = \frac{u'}{2}\sqrt{\left[\frac{m}{a}\right]^2 + \left[\frac{n}{b}\right]^2 + \left[\frac{p}{c}\right]}$$

For TM modes, $m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$, and $p = 0, 1, 2, 3, \ldots$, and for TE modes, $m = 0, 1, 2, 3, \ldots, n = 0, 1, 2, 3, \ldots$, and $p = 1, 2, 3, \ldots, m = n \neq 0$. If $a > b < c$, the dominant mode (one with the lowest resonant frequency) is $TE_{101}$.

**8.** The quality factor, a measure of the energy loss in the cavity, is given by

$$Q = \omega \frac{W}{P_L}$$

## REVIEW QUESTIONS

**12.1** At microwave frequencies, we prefer waveguides to transmission lines for transporting EM energy because of all the following *except* that

(a) Losses in transmission lines are prohibitively large.
(b) Waveguides have larger bandwidths and lower signal attenuation.
(c) Transmission lines are larger in size than waveguides.
(d) Transmission lines support only TEM mode.

**12.2** An evanscent mode occurs when

(a) A wave is attenuated rather than propagated.
(b) The propagation constant is purely imaginary.
(c) $m = 0 = n$ so that all field components vanish.
(d) The wave frequency is the same as the cutoff frequency.

**12.3** The dominant mode for rectangular waveguides is

(a) $TE_{11}$
(b) $TM_{11}$
(c) $TE_{101}$
(d) $TE_{10}$

**12.4** The $TM_{10}$ mode can exist in a rectangular waveguide.

(a) True
(b) False

**12.5** For $TE_{30}$ mode, which of the following field components exist?

(a) $E_x$
(b) $E_y$
(c) $E_z$
(d) $H_x$
(e) $H_y$

**12.6** If in a rectangular waveguide for which $a = 2b$, the cutoff frequency for $TE_{02}$ mode is 12 GHz, the cutoff frequency for $TM_{11}$ mode is

(a) 3 GHz
(b) $3\sqrt{5}$ GHz
(c) 12 GHz
(d) $6\sqrt{5}$ GHz
(e) None of the above

**12.7** If a tunnel is 4 by 7 m in cross section, a car in the tunnel will not receive an AM radio signal (e.g., $f = 10$ MHz).

(a) True
(b) False

**12.8** When the electric field is at its maximum value, the magnetic energy of a cavity is

(a) At its maximum value
(b) At $\sqrt{2}$ of its maximum value
(c) At $\frac{1}{\sqrt{2}}$ of its maximum value
(d) At 1/2 of its maximum value
(e) Zero

**12.9** Which of these modes does not exist in a rectangular resonant cavity?

(a) $TE_{110}$
(b) $TE_{011}$
(c) $TM_{110}$
(d) $TM_{111}$

**12.10** How many degenerate dominant modes exist in a rectangular resonant cavity for which $a = b = c$?

(a) 0
(b) 2
(c) 3
(d) 5
(e) $\infty$

*Answers:* 12.1c, 12.2a, 12.3d, 12.4b, 12.5b,d, 12.6b, 12.7a, 12.8e, 12.9a, 12.10c.

## PROBLEMS

**12.1** (a) Show that a rectangular waveguide does not support $TM_{10}$ and $TM_{01}$ modes.
(b) Explain the difference between $TE_{mn}$ and $TM_{mn}$ modes.

**12.2** A 2-cm by 3-cm waveguide is filled with a dielectric material with $\varepsilon_r = 4$. If the waveguide operates at 20 GHz with $TM_{11}$ mode, find: (a) cutoff frequency, (b) the phase constant, (c) the phase velocity.

**12.3** A 1-cm $\times$ 2-cm waveguide is filled with deionized water with $\varepsilon_r = 81$. If the operating frequency is 4.5 GHz, determine: (a) all possible propagating modes and their cutoff frequencies, (b) the intrinsic impedance of the highest mode, (c) the group velocity of the lowest mode.

**12.4** Design a rectangular waveguide with an aspect ratio of 3 to 1 for use in the k band (18–26.5 GHz). Assume that the guide is air filled.

**12.5** A tunnel is modeled as an air-filled metallic rectangular waveguide with dimensions $a = 8$ m and $b = 16$ m. Determine whether the tunnel will pass: (a) a 1.5-MHz AM broadcast signal, (b) a 120-MHz FM broadcast signal.

**12.6** In an air-filled rectangular waveguide, the cutoff frequency of a $TE_{10}$ mode is 5 GHz, whereas that of $TE_{01}$ mode is 12 GHz. Calculate

(a) The dimensions of the guide
(b) The cutoff frequencies of the next three higher TE modes
(c) The cutoff frequency for $TE_{11}$ mode if the guide is filled with a lossless material having $\varepsilon_r = 2.25$ and $\mu_r = 1$.

**12.7** An air-filled hollow rectangular waveguide is 150 m long and is capped at the end with a metal plate. If a short pulse of frequency 7.2 GHz is introduced into the input end of the guide, how long does it take the pulse to return to the input end? Assume that the cutoff frequency of the guide is 6.5 GHz.

**12.8** Calculate the dimensions of an air-filled rectangular waveguide for which the cutoff frequencies for $TM_{11}$ and $TE_{03}$ modes are both equal to 12 GHz. At 8 GHz, determine whether the dominant mode will propagate or evanesce in the waveguide.

**12.9** An air-filled rectangular waveguide has cross-sectional dimensions $a = 6$ cm and $b = 3$ cm. Given that

$$E_z = 5 \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right) \cos\left(10^{12}t - \beta z\right) \text{ V/m}$$

calculate the intrinsic impedance of this mode and the average power flow in the guide.

**12.10** In an air-filled rectangular waveguide, a TE mode operating at 6 GHz has

$$E_y = 5 \sin(2\pi x/a) \cos(\pi y/b) \sin(\omega t - 12z) \text{ V/m}$$

Determine: (a) the mode of operation, (b) the cutoff frequency, (c) the intrinsic impedance, (d) $H_x$.

**12.11** In an air-filled rectangular waveguide with $a = 2.286$ cm and $b = 1.016$ cm, the $y$-component of the TE mode is given by

$$E_y = \sin(2\pi x/a)\cos(3\pi y/b)\sin(10\pi \times 10^{10}t - \beta z) \text{ V/m}$$

find: (a) the operating mode, (b) the propagation constant $\gamma$, (c) the intrinsic impedance $\eta$.

**12.12** For the $TM_{11}$ mode, derive a formula for the average power transmitted down the guide.

**12.13** (a) Show that for a rectangular waveguide.

$$u_p = \frac{u'}{\sqrt{1 - \left[\frac{f_c}{f}\right]^2}} \qquad \lambda = \frac{\lambda'}{\sqrt{1 - \left[\frac{f_c}{f}\right]^2}}$$

(b) For an air-filled waveguide with $a = 2b = 2.5$ cm operating at 20 GHz, calculate $u_p$ and $\lambda$ for $TE_{11}$ and $TE_{21}$ modes.

**12.14** A 1-cm $\times$ 3-cm rectangular air-filled waveguide operates in the $TE_{12}$ mode at a frequency that is 20% higher than the cutoff frequency. Determine: (a) the operating frequency, (b) the phase and group velocities.

**12.15** A microwave transmitter is connected by an air-filled waveguide of cross section 2.5 cm $\times$ 1 cm to an antenna. For transmission at 11 GHz, find the ratio of (a) the phase velocity to the medium velocity, and (b) the group velocity to the medium velocity.

**12.16** A rectangular waveguide is filled with polyethylene ($\varepsilon = 2.25\varepsilon_o$) and operates at 24 GHz. If the cutoff frequency of a certain TE mode is 16 GHz, find the group velocity and intrinsic impedance of the mode.

**12.17** A rectangular waveguide with cross sections shown in Figure 12.16 has dielectric discontinuity. Calculate the standing wave ratio if the guide operates at 8 GHz in the dominant mode.

***12.18** Analysis of circular waveguide requires solution of the scalar Helmholtz equation in cylindrical coordinates, namely

$$\nabla^2 E_{zs} + k^2 E_{zs} = 0$$

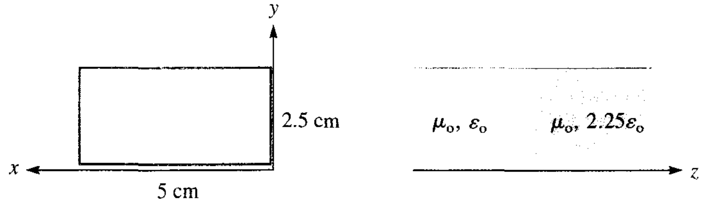


Figure 12.16 For Problem 12.17.

or

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial E_{zs}}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_{zs}}{\partial \phi^2} + \frac{\partial^2 E_{zs}}{\partial z^2} + k^2 E_{zs} = 0$$

By assuming the product solution

$$E_{zs}(\rho, \phi, z) = R(\rho)\,\Phi(\phi)\,Z(z)$$

show that the separated equations are:

$$Z'' - k_z^2 Z = 0$$

$$\Phi'' + k_\phi^2 \Phi = 0$$

$$\rho^2 R'' + \rho R' + (k_\rho^2 \rho^2 - k_\phi^2) R = 0$$

where

$$k_\rho^2 = k^2 + k_z^2$$

**12.19** For $\text{TE}_{01}$ mode,

$$E_{xs} = \frac{j\omega\mu\pi}{bh^2} H_o \sin(\pi y/b) e^{-\gamma z}, \qquad E_{ys} = 0$$

Find $\mathcal{P}_{\text{ave}}$ and $P_{\text{ave}}$.

**12.20** A 1-cm × 2-cm waveguide is made of copper ($\sigma_c = 5.8 \times 10^7$ S/m) and filled with a dielectric material for which $\varepsilon = 2.6\varepsilon_o$, $\mu = \mu_o$, $\sigma_d = 10^{-4}$ S/m. If the guide operates at 9 GHz, evaluate $\alpha_c$ and $\alpha_d$ for (a) $\text{TE}_{10}$, and (b) $\text{TM}_{11}$.

**12.21** A 4-cm-square waveguide is filled with a dielectric with complex permittivity $\varepsilon_c = 16\varepsilon_o(1 - j10^{-4})$ and is excited with the $\text{TM}_{21}$ mode. If the waveguide operates at 10% above the cutoff frequency, calculate attenuation $\alpha_d$. How far can the wave travel down the guide before its magnitude is reduced by 20%?

**12.22** If the walls of the square waveguide in the previous problem are made of brass ($\sigma_c = 1.5 \times 10^7$ S/m), find $\alpha_c$ and the distance over which the wave is attenuated by 30%.

**12.23** A rectangular waveguide with $a = 2b = 4.8$ cm is filled with teflon with $\varepsilon_r = 2.11$ and loss tangent of $3 \times 10^{-4}$. Assume that the walls of the waveguide are coated with gold ($\sigma_c = 4.1 \times 10^7$ S/m) and that a $\text{TE}_{10}$ wave at 4 GHz propagates down the waveguide, find: (a) $\alpha_d$, (b) $\alpha_c$.

***12.24** A rectangular brass ($\sigma_c = 1.37 \times 10^7$ S/m) waveguide with dimensions $a = 2.25$ cm and $b = 1.5$ cm operates in the dominant mode at frequency 5 GHz. If the waveguide is filled with teflon ($\mu_r = 1$, $\varepsilon_r = 2.11$, $\sigma \simeq 0$), determine: (a) the cutoff frequency for the dominant mode, (b) the attenuation constant due to the loss in the guide walls.

***12.25** For a square waveguide, show that attenuation $\alpha_c$ is minimum for $\text{TE}_{10}$ mode when $f = 2.962 f_c$.

**12.26** The attenuation constant of a TM mode is given by

$$\alpha = \frac{2}{\eta_o}\sqrt{\frac{\pi f \mu/\sigma}{1 - \left(\frac{f_c}{f}\right)}}$$

At what frequency will $\alpha$ be maximum?

***12.27** Show that for TE mode to $z$ in a rectangular cavity,

$$E_{ys} = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{c}\right)$$

Find $H_{xs}$.

***12.28** For a rectangular cavity, show that

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right)E_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)$$

for TM mode to $z$. Determine $E_{ys}$.

**12.29** In a rectangular resonant cavity, which mode is dominant when

(a) $a < b < c$
(b) $a > b > c$
(c) $a = c > b$

**12.30** For an air-filled rectangular cavity with dimensions $a = 3$ cm, $b = 2$ cm, $c = 4$ cm, determine the resonant frequencies for the following modes: $TE_{011}$, $TE_{101}$, $TM_{110}$, and $TM_{111}$. List the resonant frequencies in ascending order.

**12.31** A rectangular cavity resonator has dimensions $a = 3$ cm, $b = 6$ cm, and $c = 9$ cm. If it is filled with polyethylene ($\varepsilon = 2.5\varepsilon_o$), find the resonant frequencies of the first five lowest-order modes.

**12.32** An air-filled cubical cavity operates at a resonant frequency of 2 GHz when excited at the $TE_{101}$ mode. Determine the dimensions of the cavity.

**12.33** An air-filled cubical cavity of size 3.2 cm is made of brass ($\sigma_c = 1.37 \times 10^7$ S/m). Calculate: (a) the resonant frequency of the $TE_{101}$ mode, (b) the quality factor at that mode.

**12.34** Design an air-filled cubical cavity to have its dominant resonant frequency at 3 GHz.

**12.35** An air-filled cubical cavity of size 10 cm has

$$\mathbf{E} = 200 \sin 30\pi x \sin 30\pi y \cos 6 \times 10^9 t\, \mathbf{a}_z \text{ V/m}$$

Find **H**.