Chapter 2

Transmission Lines

Contents

2.1 Introduction

In this second chapter your knowledge of circuit theory is connected into the study transmission lines having voltage and current along the line in terms of 1D traveling waves. The transmission line is a *two-port* circuit used to connect a generator or transmitter signal to a receiving load over a distance. In simple terms power transfer takes place.

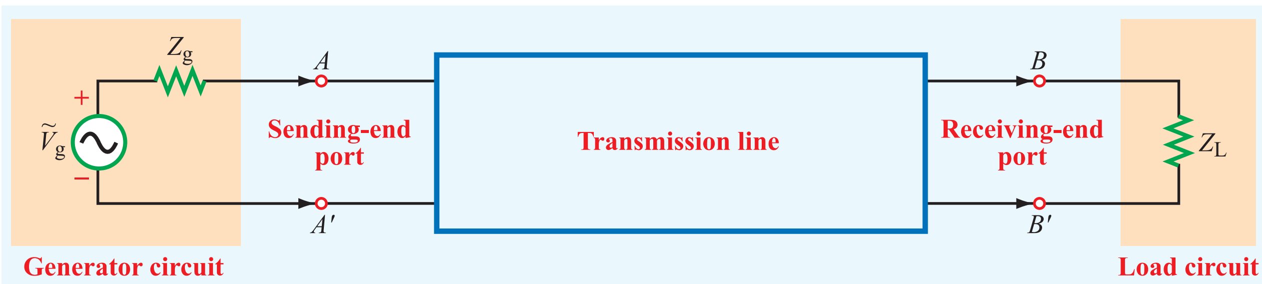

Figure 2.1: Two-port model of a transmission line.

- In the beginning the transmission line is developed as a lumped element circuit, but then a limit is taken to convert the circuit model into a *distributed element* circuit
  - Distributed element means that element values such as R, L, and C become R, L, and C per unit length of the line, e.g, Ω/m, H/m, and F/m respectively
- In the lumped element model we have a pair of differential equations to describe the voltage and current along the line
- Under the limiting argument which converts the circuit to distributed element form, we now have a pair of *partial* differential equations, which when solved yield a solution that is a 1D traveling wave

- Key concepts developed include: wave propagation, standing waves, and power transfer

- Returning to Figure 2.1, we note that sinusoidal steady-state is implied as the source voltage is the phasor $\tilde{V}_g$, the source impedance Z_g is a Thévenin equivalent and Z_L is the load impedance

  – The source may be an RF transmitter connecting to an antenna

  – The source/load may be a pair at each end of the line when connecting to an Ethernet hub in wired a computer network

  – The source may be power collected by an antenna and via cable to the receiver electronics, e.g., satellite TV (*Dish Network*)

2.1.1 The Role of Wavelength

- When circuits are interconnected with wires (think protoboard or a printed circuit board (PCB)), is a transmission line present?

- The answer is **yes**, for better or worse

- As long the circuit interconnect lengths are *small* compared with the wavelength of the signals present in the circuit, lumped element circuit characteristics prevail

  – Ulaby suggests transmission line effects may be ignored when $l/\lambda \lesssim 0.01$ (I have been content with a $\lambda/10$ limit)

– Lumped element capacitance and inductance (parasitics) due to interconnects may still alter circuit performance

Power Loss and Dispersion

- Transmission lines may also be *dispersive*, which means the propagation velocity on the line is not constant with frequency
- For example the frequency components of square wave (recall odd harmonics only) each propagate at a different velocity, meaning the waveform becomes *smeared*
- Dispersion is very important to high speed digital transmission (fiber optic and wired networks alike)
- The longer the line, the greater the impact

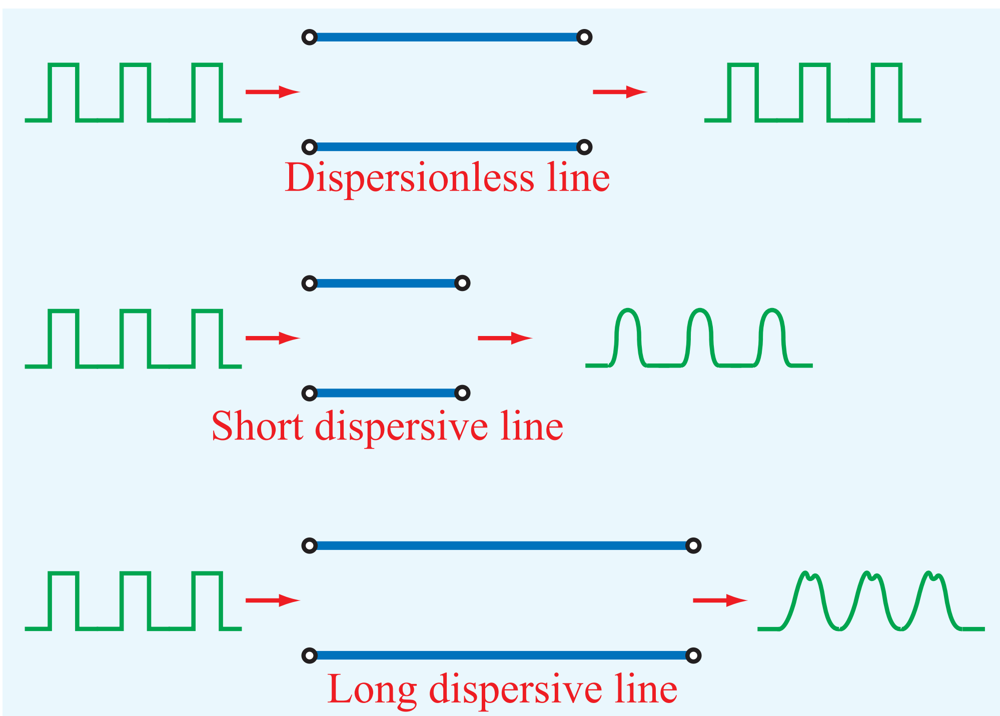

Figure 2.2: The impact of transmission line dispersion.

2.1.2 Propagation Modes

- When a time-varying signal such as sinusoid connected to (or launched on) a transmission line, a propagation *mode* is established
- Recall that both electric and magnetic fields will be present (why?)
- Two mode types as: (1) *transvere electromagnetic (TEM)* and (2) non-TEM or *higher-order*

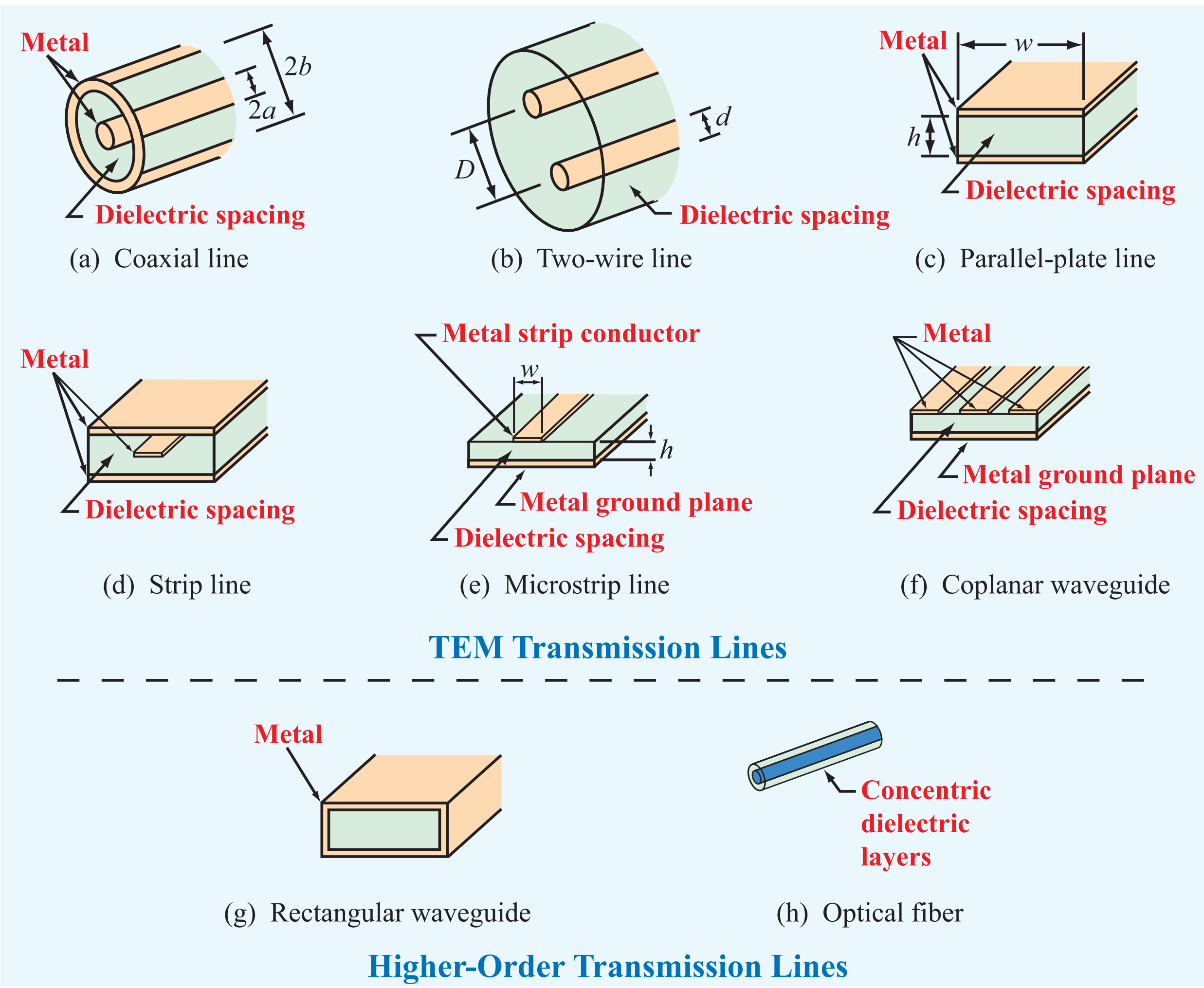

Figure 2.3: A collection classical TEM transmission lines (a)–(c), mircowave circuit TEM lines (d)–(f), and non-TEM lines (g) & (h).

- TEM, which means electric and magnetic fields are *transverse* to the direction of propagation, is the exclusive study of Chapter 2

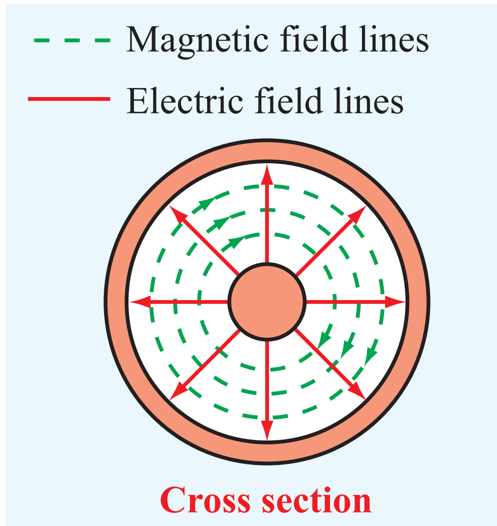

Figure 2.4: The transverse fields of the coax.

- non-TEM means one or more field components lies in the direction of propagation, e.g. metal or optical fiber waveguides

- We start with a *lumped element* model of a TEM line and derive the *telegrapher's equations*

2.2 The Lumped Element Model

- TEM transmission lines all exhibit *axial symmetry*

- From a circuit theory perspective the line is completely described by four *distributed parameter* quantities:

  – R': The effective series resistance per unit length in Ω/m (all conductors that make up the line cross-section are included)

- L': The effective series inductance per unit length in H/m (again all conductors that make up the line cross-section are included)
- G': The effective shunt conductance of the line insulation (air and/or dielectric) per unit length in S/m (mhos/m?)
- C': The effective shunt capacitance per unit length between the line conductors in F/m

- We will see that these four parameters are calculated from (1) the line cross-sectional geometry (see 2.4) and (2) the EM constitutive parameters (μ_c and σ_c for conductors and ϵ, μ, and σ insulation/dielectric)

- For the three classical line types the four parameters are calculated using the table below (also see the text *Java modules*[1])

Table 2.1: R', L', G', and C' for the four classical line types.

Parameter	**Coaxial**	**Two-Wire**	**Parallel-Plate**	**Unit**
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2R_s}{\pi d}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]$	$\frac{\mu h}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\sigma w}{h}$	S/m
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\varepsilon w}{h}$	F/m

Notes: (1) Refer to **Fig. 2-4** for definitions of dimensions. (2) μ, ε, and σ pertain to the insulating material between the conductors. (3) $R_s = \sqrt{\pi f \mu_c/\sigma_c}$. (4) μ_c and σ_c pertain to the conductors. (5) If $(D/d)^2 \gg 1$, then $\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right] \approx \ln(2D/d)$.

[1] http://em7e.eecs.umich.edu/ulaby_modules.html

2.2.1 Coax Detail

- The formula for R' (see text Chapter 7) is

$$R' = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \quad (\Omega/\mathrm{m})$$

  where R_s is the effective *surface resistance* of the line

- In text Chapter 7 it is shown that

$$R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}} \quad (\Omega)$$

  - **Note**: The series resistance of the line increases as the $\sqrt{f}$
  - **Also Note**: If $\sigma_c \gg f\mu_c \; R' \simeq 0$

- The formula for L' (see text Chapter 5) is

$$L' = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) \quad (\mathrm{H/m})$$

  where is the joint inductance of both conductors in the line cross-section

- The formula for G' (see text Chapter 4) is given by

$$G' = \frac{2\pi\sigma}{\ln(b/a)} \quad (\mathrm{S/m})$$

- **Note**: The non-zero conductivity of the line allows current to flow between the conductors (lossless dielectric $\rightarrow \sigma = 0$ and $G' = 0$)

- The formula for C' (see Chapter 4), which is the shunt capacitance between the conductors, is

$$C' = \frac{2\pi\epsilon}{\ln(b/a)} \quad \text{F/m}$$

- The voltage difference between the conductors sets up equal and opposite charges and it is the ratio of the charge to voltage difference that defines the capacitance

2.2.2 TEM Line Facts

- Velocity of propagation relationship:

$$\boxed{L'C' = \mu\epsilon \Rightarrow u_p = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{L'C'}}}$$

  - Recall from Chapter 1 $c = 1/\sqrt{\mu_0\epsilon_0}$

- Secondly,

$$\boxed{\frac{G'}{C'} = \frac{\sigma}{\epsilon}}$$

Example 2.1: **Coax Parameter Calculation Using Module 2.2**

- A coaxial air line is operating at 1 MHz

- The inner radius is $a = 6$ mm and the outer radius is $b = 12$ mm

- Assume copper for the conductors ($\mu \approx 1$)

- Use the text Module 2.2 to obtain the transmission line parameters

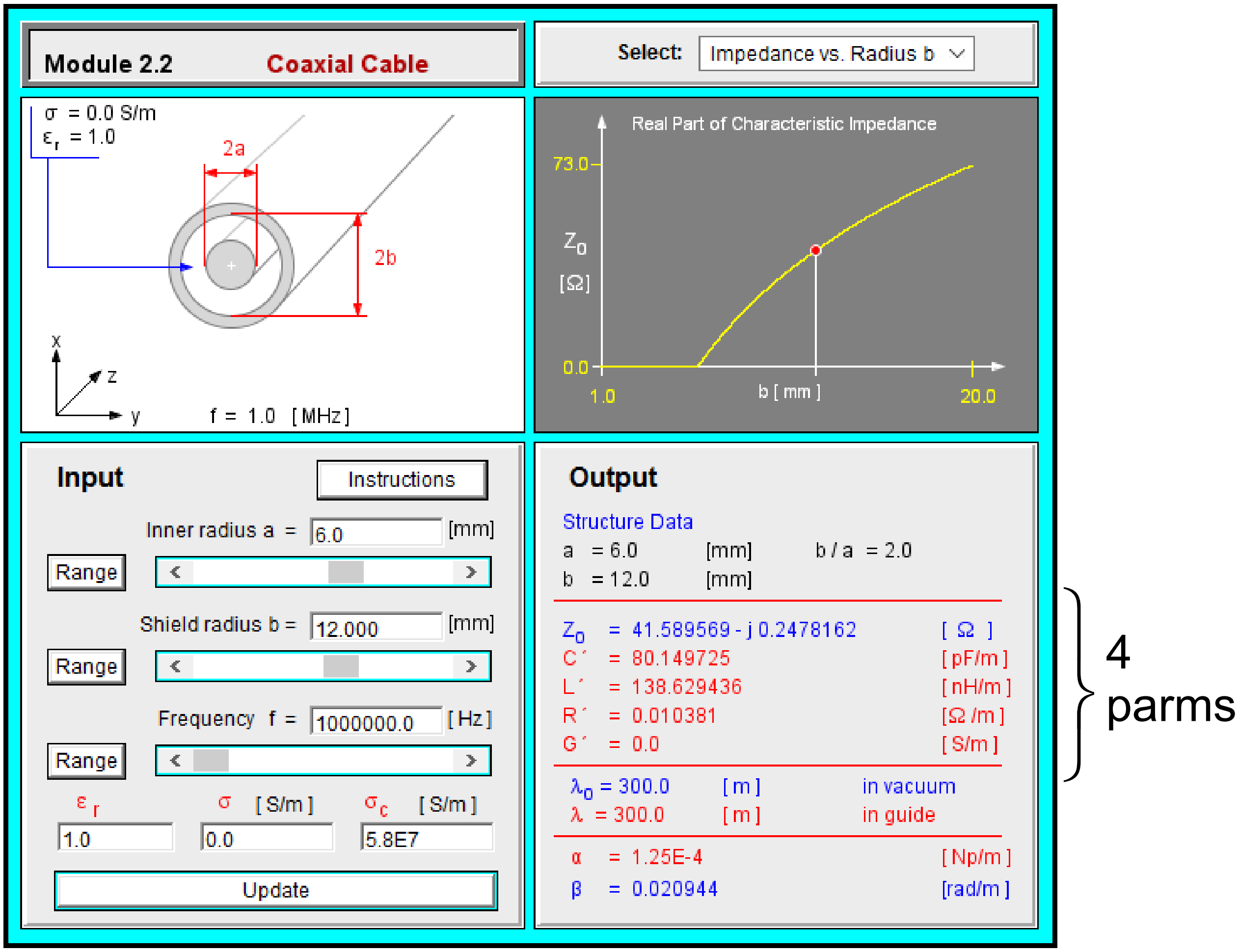

Figure 2.5: Transmission line parameters obtained from text Module 2.2

- As an alternative, it is not much work to do the same calculation in the Jupyter notebook

- With hand coding many options exist to do more than just obtain R', L', G', and C'

- Soon we will see other tline characteristics derived from the four distributed element parameters

```
def coax_parameters(a,b,f,e_r,sigma_c,sigma=1,mu=1,mu_c=1,display=True):
    """
    Find the distributed parameters of the coax line

    R_p, L_p, G_p, C_p = coax_parameters(a,b,f,e_r,sigma_c,
                                         sigma=1,mu=1,mu_c=1,
                                         display=True)
    Note: By default display is True, so to supress raw variables
          use a semicolon to terminate the function call.

    Mark Wickert February 2016
    """
    e_0 = 8.85e-12 # F/m
    mu_0 = 4*pi*1e-7
    Rs = sqrt(pi*f*mu_0*mu_c/sigma_c)
    R_p = Rs/(2*pi)*(1/a +1/b)  # Ohms/m
    L_p = mu_0*mu/(2*pi)*log(b/a)  #H/m
    G_p = 2*pi*sigma/log(b/a)  # S/m
    C_p = 2*pi*e_0*e_r/log(b/a)
    # Display calculations as opposed to returning the value
    if display:
        print("R' = %2.5f ohms/m" % (R_p,))
        print("L' = %2.2f nH/m" % (L_p*1e9,))
        print("G' = %2.3f S/m" % (G_p,))
        print("C' = %2.2f pF/m" % (C_p*1e12,))
    return R_p, L_p, G_p, C_p
```

```
coax_parameters(6e-3,12e-3,1e6,1.0,5.8e7);
```

```
R' = 0.01038 ohms/m
L' = 138.63 nH/m
G' = 9.065 S/m
C' = 80.22 pF/m
```

Figure 2.6: Transmission line parameters from Python function `coax_parameters()`

2.3 The Transmission Line Equations

- The moment you have been waiting for: solving the tline equations starting from a differential length of line (classical!)

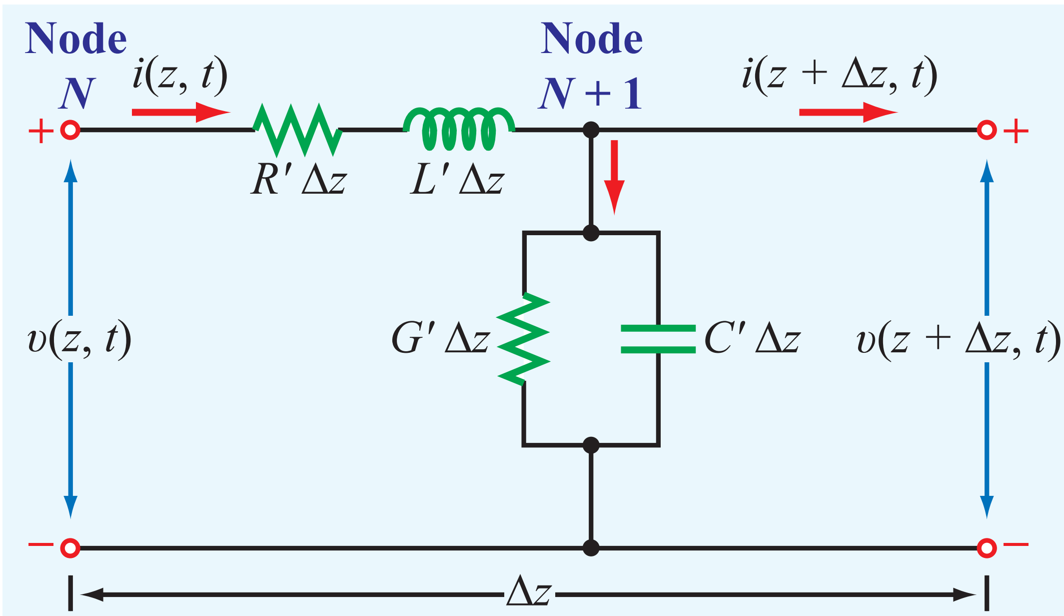

Figure 2.7: A differential section of TEM transmission line, Δz ready for Kirchoff's laws.

- The goal of this section is to obtain the voltage across the line $v(z,t)$ and the current through the line $i(z,t)$ for any $-l \leq z < 0$ and t value
  - **Note**: It is customary to let the line length be l and have $z = 0$ at the load end and $z = -l$ at the source end (more later)
- From Kirchoff's voltage law we sum voltage drops around the loop to zero:

$$v(z,t) - R'\Delta z\, i(z,t) - L'\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0$$

- Getting set for form a derivative in the limit, we divide by Δz everywhere are rearrange

$$-\left[\frac{v(z+\Delta z,t) = v(z,t)}{\Delta z}\right] = R'i(z,t) + L'\frac{\partial i(z,t)}{\partial t}$$

- Taking the limit as $\Delta z \to 0$ gives

$$\boxed{-\frac{\partial v(z,t)}{\partial z} = R'i(z,t) + L'\frac{\partial i(z,t)}{\partial t}}$$

- From Kirchoff's current law we sum current entering minus current leaving node $N+1$ to zero:

$$\begin{aligned} &i(z,t) - G'\Delta z\, v(z+\Delta z,t) - C'\Delta z\,\frac{\partial v(z+\delta z,t)}{\partial t} \\ &\quad - i(z+\Delta z,t) = 0 \end{aligned}$$

- Rearranging in similar fashion to the voltage equation, and taking the limit as $\Delta z \to 0$, yields

$$\boxed{-\frac{\partial i(z,t)}{\partial z} = G'v(z,t) + C'\frac{\partial v(z,t)}{\partial t}}$$

- The above blue-boxed equations are the *telegrapher's equations in the time domain*
  - A full time-domain solution is available, but will be deferred to later in the chapter (I like personally like starting here)

- A sinusoidal steady-state solution (frequency domain) is possible to if you let

$$\begin{aligned} v(z,t) &= \mathrm{Re}\big[\tilde{V}(z)e^{j\omega t}\big] \\ i(z,t) &= \mathrm{Re}\big[\tilde{I}(z)e^{j\omega t}\big], \end{aligned}$$

with $\tilde{V}(z)$ and $\tilde{I}(z)$ being the phasor components corresponding to $v(z,t)$ and $i(z,t)$ respectively

- The phasor form of the telegrapher's equations is simply

$$-\frac{d\tilde{V}(z)}{dz} = \left(R' + j\omega L'\right)\tilde{I}(z)$$
$$-\frac{d\tilde{I}(z)}{dz} = \left(G' + j\omega C'\right)\tilde{V}(z)$$

Note: $\partial \to d$ since t is suppressed in the phasor form

- Solving the 1D wave equation is next

2.4 Wave Propagation on a Transmission Line

- The phasor telegrapher equations are coupled, that is one equation can be inserted into the other to eliminate either $\tilde{V}(z)$ or $\tilde{I}(z)$ and in return have a second-order differential equation in z

2.4.1 Getting to the 1D Wave Equation

- To eliminate $\tilde{I}(z)$ differentiate the first equation with respect to z so that $d\tilde{I}(z)/dz$ is found in both equations

$$\frac{d^2\tilde{V}(z)}{dz^2} = \left(R' + j\omega L'\right)\frac{d\tilde{I}(z)}{dz},$$

then replace $d\tilde{I}(z)/dz$ with the right side of the second equation

$$\frac{d^2\tilde{V}(z)}{dz^2} - \big(R' + j\omega L'\big)\big(G' + j\omega C'\big)\tilde{V}(z) = 0$$

or in the form of the phasor-based 1D voltage wave equation

$$\boxed{\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0,}$$

where γ is the complex *propagation constant*

$$\boxed{\gamma = \sqrt{\big(R' + j\omega L'\big)\big(G' + j\omega C'\big)}}$$

- Working with the telegrapher's equations to solve for $\tilde{I}(z)$ results in the current wave equation

$$\boxed{\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0}$$

- **Note**: When γ is expanded out it becomes

$$\gamma = \alpha + j\beta$$

where as seen in Chapter 1, α is the wave attenuation constant in Np/m and β is the phase constant, which (recall) equals $2\pi/\lambda$

- In full detail

$$\alpha = \mathrm{Re}\left(\sqrt{\big(R' + j\omega L'\big)\big(G' + j\omega C'\big)}\right) \quad \text{(Np/m)}$$

$$\beta = \mathrm{Re}\left(\sqrt{\big(R' + j\omega L'\big)\big(G' + j\omega C'\big)}\right) \quad \text{(rad/m)}$$

- For low loss lines the equations for α and β simplify (more later)
- For now just know that for a passive tline $\alpha \geq 0$

2.4.2 Solving the Wave Equation (Phasor Form)

- The general solutions of the two wave equations each involve a pair of exponentials (think back to 2nd-order differential equations)

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \text{(V)}$$
$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \quad \text{(A)}$$

where it becomes clear later that $e^{-\gamma z}$ is a wave propagating in the $+z$ direction and $e^{\gamma z}$ is a wave propagating in the $-z$ direction

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Example 2.2: **Verify the General Solution Satisfies Wave Equation**

- Is it really true that $\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$ satisfies $d^2\tilde{V}(z)/dz^2 - \gamma^2\tilde{V}(z) = 0$?
- Form $d^2\tilde{V}(z)/z^2$ and see:

$$\frac{d}{dz}\tilde{V}(z) = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z}$$
$$\frac{d^2}{dz^2}\tilde{V}(z) = \gamma^2 V_0^+ e^{-\gamma z} + \gamma^2 V_0^- e^{\gamma z} \stackrel{\text{also}}{=} \gamma^2\tilde{V}(z) \quad \checkmark$$

- The variables (V_0^+, I_0^+) and (V_0^-, I_0^-) are unknowns that will be found using *boundary conditions*

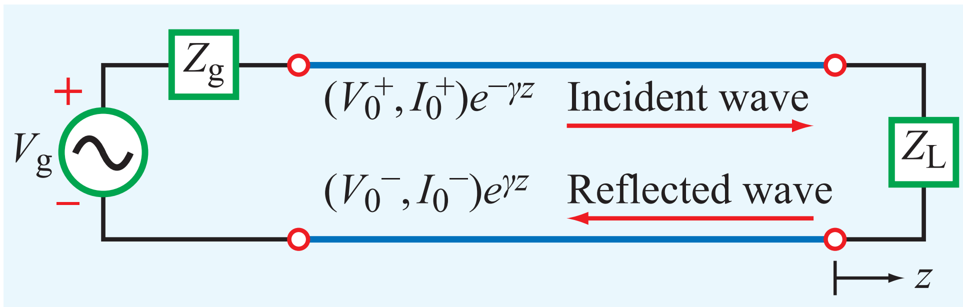


Figure 2.8: Incident and reflected traveling waves on a tline.

- How do you know the solution works?
- Plug it into the voltage and current wave equations and see
- You can eliminate two (I_0^+ and I_0^-) of the four unknowns by inserting the general solution for $\tilde{V}(z)$ into the first of the telegrapher's equations, i.e.,

$$-\frac{\tilde{V}(z)}{dz} = -\frac{d}{dz}\left[V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}\right] = \gamma\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right]$$
$$\stackrel{\text{also}}{=} \left(R' + j\omega L'\right)\tilde{I}(z),$$

so solving for $\tilde{I}(z)$ yields

$$\tilde{I}(z) = \frac{\gamma}{\left(R' + j\omega L'\right)}\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right]$$
$$\stackrel{\text{also}}{=} I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

where the last line follows from the second general wave equation solution

- What just happened? Confused?

- It appears that

$$\frac{V_0^+}{I_0^+} \overset{\textbf{must}}{=} \frac{(R' + j\omega L')}{\gamma} \overset{\textbf{must}}{=} -\frac{V_0^-}{I_0^-}$$

- We define

$$\boxed{Z_0 = \frac{(R' + j\omega L')}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad (\Omega)}$$

Note: The units look good as V/I is R from Ohm's law

- Furthermore, Z_0 involves the ratio of incident or reflected voltage over current waves, not the total voltage and current, $\tilde{V}(z)$ and $\tilde{I}(z)$

- **Summarizing**: The general voltage and current solutions are now down to just two unknowns

$$\boxed{\begin{aligned} \tilde{V}(z) &= V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad (\text{V}) \\ \tilde{I}(z) &= \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \quad (\text{A}) \end{aligned}}$$

- Secondly, Z_0 although clearly a function of ω, it is the line cross-sectional geometry and constitutive parameters that really control its value

- Finally, when source Z_g and load Z_L boundary conditions are applied, you will be able to solve for the complex quantities V_0^+ and V_0^-

2.4.3 Returning to the Time Domain

- For the sinusoidal steady-state solution, a return to the time domain is possible by writing $V_0^+ = |V_0^+|e^{j\phi^+}$ and $V_0^- = |V_0^-|e^{j\phi^+}$, then plug into the cosine form:

$$\begin{aligned} v(z,t) &= \mathrm{Re}\big[\tilde{V}(z)e^{j\omega t}\big] \\ &= |V_0^+|e^{-\alpha z}\cos\big(\omega t - \beta z + \phi^+\big) \\ &\quad + |V_0^-|e^{\alpha z}\cos\big(\omega t + \beta z + \phi^-\big) \end{aligned}$$

- When you consider LTspice simulations using `.tran` a little bit later, this is what you will be observing

- Also, the first term travels in the $+z$ direction while the second term travels in the $-z$ direction

- The velocity of propagation for both waves is of course

$$u_p = f\lambda = \frac{\omega}{\beta}$$

- Observe that just as in HMWK problem 1.8, a *standing wave* pattern results when the two propagating waves superimpose

- Solving for V_0^+ and V_0^- is still an open problem

Example 2.3: **No Specific Geometry Air Line**

- In this example we assume that the two conductors of a transmission line are in air

- Furthermore, the air assumption is taken to mean $\sigma = 0$

- The conductors are perfect, that is $R_s = 0$
- The requirements are to find L' and C', given that $Z_0 = 50\ \omega$, $\beta = 20$ rad/m, and the operating frequency is $f = 700$ MHz
- The first set of assumptions tell us that $R' = G' = 0$
- The expressions for β and Z_0 in terms of the tline distributed element parameters, simplify greatly under the lossless conditions

$$\beta = \mathrm{Im}\left[\sqrt{(0 + j\omega L')(0 + j\omega C')}\right] = \omega\sqrt{L'C'}$$

$$Z_0 = \sqrt{\frac{0 + j\omega L'}{0 + j\omega C'}} = \sqrt{\frac{L'}{C'}},$$

  so with two equations involving the unknowns L' and C' we can solve for them using the Z_0, β, and f

- Start by forming β/Z_0

$$\frac{\beta}{Z_0} = \omega C' = 2\pi \cdot f \cdot C',$$

  which implies that

$$C' = \frac{\beta}{2\pi f \cdot Z_0} = \frac{20}{2\pi \cdot 7 \times 10^8 \cdot 50} = \underbrace{9.09 \times 10^{-11} \quad \text{(F/m)}}_{90.9\ \text{(pF/m)}}$$

- With C' found, the Z_0 expression can be used to find L'

$$L' = Z_0^2 C' = 50^2 \cdot 9.09 \times 10^{-11} = \underbrace{2.27 \times 10^{-7} \quad \text{(H/m)}}_{227\ \text{(nH/m)}}$$

2.5 The Lossless Microstrip Line

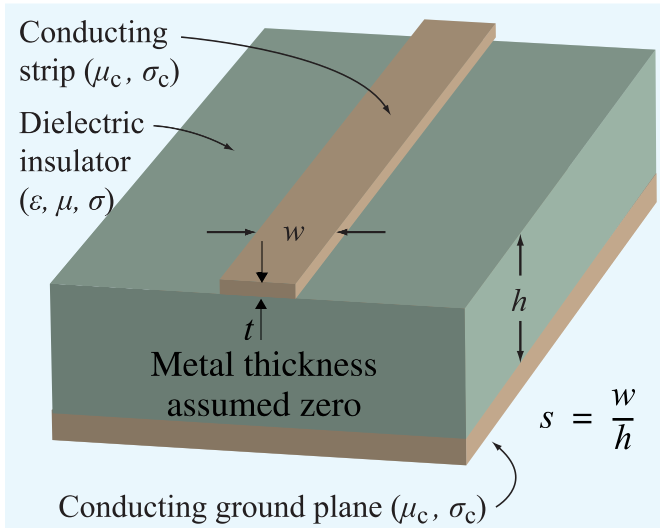


Table 2.2: Microstrip geometry.

- In RF/microwave circuit design the microstrip structure of Figure 2.3e and also shown immediately above, is very common
- The tline geometry fits well with surface-mount PCB design of today
  - A narrow strip of width w sits on top of a dielectric substrate of height h, which in turn sits over a ground plane
  - Surface-mount components can be mounted on the top of the substrate
- A downside of microstrip is that the air-dielectric interface gives rise to a small axial field component, making the propagation mode *quasi-TEM*
- The nature of quasi-TEM is that the line introduces some dispersion (recall this means constant u_p versus frequency)

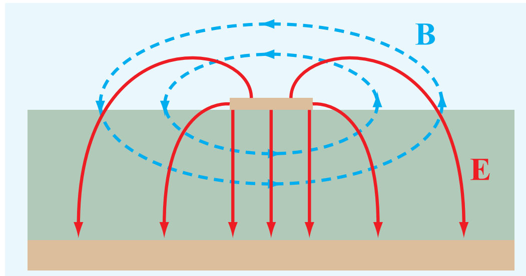


Figure 2.9: Electric and Magnetic fields of mircostrip

- In the discussion and analysis of the text, the dielectric filling material is assumed to be lossless ($\sigma = 0$)
- The conductors, strip on top and ground plane on the bottom, are assumed perfect ($\sigma \approx \infty$)
- Finally, the permeability is $\mu = \mu_0$, that is the material is nonmagnetic
  - Note there are applications involving microstrip where magnetic materials are used, just not discussed here

2.5.1 Analysis and Synthesis of Microstrip

- For tlines with a homogeneous dielectric filling (also nonmagnetic), e.g., coax and stripline of Figure 2.3, the velocity of wave propagation or phase velocity, is always

$$u_p \overset{\text{also}}{=} v_p = \frac{1}{\sqrt{\mu_0 \epsilon_0 \epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}}$$

- Considering Figure 2.9, the electric field lines are mostly in the dielectric, but some are topside in *air*

- In the analysis of microstrip the mixture of air and dielectric is managed by defining ϵ_{eff}, the *effective relative permittivity*, hence the phase velocity is written as

$$\boxed{u_p = v_p = \frac{c}{\sqrt{\epsilon_{\text{eff}}}}}$$

- The detailed analysis of microstrip can be found in text books such as[2]

- There are three basic equations you need to be familiar with when working with microstrip:

  1. An equation that finds ϵ_{eff} given the strip aspect ratio $s = w/h$ and ϵ_r
  2. An equation that takes s together with ϵ_{eff} to find the line characteristic impedance Z_0
  3. An equation that takes Z_0 together with ϵ_r to find $s = w/h$; generally ϵ_{eff} is also found along the way

- In *microwave circuit design* various Z_0 values will needed along with the corresponding ϵ_{eff} values $\Rightarrow$ (3) is very valuable!

- The equations are *empirical* function fits to detailed field analysis results for tline inductance and capacitance per unit length, as aspect ratio, s, changes

[2] D. H. Schrader, *Microwave Circuit Analysis, Prentice Hall, 1995*

Obtain ϵ_{eff} From w/h and ϵ_r

- The empirical formula for zero thickness strips ($t/h = 0$) is given in terms of $s = w/h$ by

$$\epsilon_{\text{eff}} = \frac{e_r + 1}{2} + \left(\frac{\epsilon_r - 1}{2}\right)\left(1 + \frac{10}{s}\right)^{-xy},$$

where

$$x = 0.56\left[\frac{\epsilon_r - 0.9}{\epsilon_r + 3}\right]$$
$$y = 1 + 0.02\ln\left(\frac{s^4 + 3.7\times 10^{-4}s^2}{s^4 + 0.43}\right) + 0.05\ln(1 + 1.7\times 10^{-4}s^3)$$

- Chapter 2 notebook Python function implementation:

```python
def eps_eff(s,er):
    """
    Find microstrip e_eff given s = w/h and er

    Mark Wickert February 2016
    """
    x = 0.56*((er-0.9)/(er+3))**0.05
    y = 1 + 0.02*log((s**4 + 3.7e-4*s**2)/(s**4+0.43)) + \
        0.05*log(1+1.7e-4*s**3)
    e_eff = (er+1)/2 + (er-1)/2*(1+10/s)**(-x*y)
    return e_eff
```

Figure 2.10: A Python function that finds ϵ_{eff} given $s = w/h$ and ϵ_r.

Obtain Z_0 From w/h and ϵ_r

- Design for zero thickness strips is given by

$$Z_0 = \frac{60}{\sqrt{\epsilon_{\text{eff}}}}\ln\left[\frac{6 + (2\pi - 6)e^{-t}}{s} + \sqrt{1 + \frac{4}{s^2}}\right],$$

where

$$t = \left(\frac{30.67}{s}\right)^{0.75}$$

- Chapter 2 notebook Python function implementation:

```
def mstrip_anal(s,er):
    """
    Microstrip zero thickness strips design function

    s = w/h
    er = material relative permittivity

    Z0 returned
    e_eff returned

    Mark Wickert February 2016
    """
    e_eff = eps_eff(s,er)
    t = (30.67/s)**0.75
    Z0 = 60/sqrt(e_eff) * \
         log((6+(2*pi-6)*exp(-t))/s + sqrt(1+4/s**2))
    return Z0, e_eff
```

Figure 2.11: A Python function that finds Z_0 given $s = w/h$ and ϵ_r.

Obtain w/h From Z_0 and ϵ_r

- Design for zero thickness strips is given by the piecewise solution

$$\frac{w}{h} = \begin{cases} \frac{2}{\pi}\Big[(q-1) - \ln(2q-1) \\ \quad +\frac{\epsilon_r - 1}{2\epsilon_r}\Big(\ln(q-1) \\ \quad +0.29 - \frac{0.52}{\epsilon_r}\Big)\Big], & Z_0 \leq (44 - 2\epsilon_r) \\ \frac{8e^p}{e^{2p}-2}, & Z_0 > (44 - 2\epsilon_r), \end{cases}$$

where

$$q = \frac{60\pi^2}{Z_0\sqrt{\epsilon_r}}$$

$$p = \sqrt{\frac{\epsilon_r + 1}{2}}\frac{Z_0}{60} + \left(\frac{\epsilon_r - 1}{\epsilon_r + 1}\right)\left(0.23 + \frac{0.12}{\epsilon_r}\right)$$

- Chapter 2 notebook Python function implementation:

```
def mstrip_dsgn(Z0,er):
    """
    Mircostrip zero thickness strips design function

    Z0 = desired characteristic impedance
    er = material relative permittivity

    s = w/h is returned
    e_eff returned

    Mark Wickert February 2016
    """
    if Z0 <= (44-2*er):
        q = 60*pi*pi/(Z0*sqrt(er))
        s = 2/pi*((q-1)-log(2*q-1) + \
                  (er-1)/(2*er)*(log(q-1)+0.29-0.52/er))

    else:
        p = sqrt((er+1)/2)*Z0/60+(er-1)/(er+1)*(0.23 + 0.12/er)
        s = 8*exp(p)/(exp(2*p)-2)
    e_eff = eps_eff(s,er)
    return s, e_eff
```

Figure 2.12: A Python function that finds $s = w/h$ given Z_0 and ϵ_r.

2.5.2 Common Mstrip Materials

- **FR-4** made of epoxy fiberglass with a fire retardant property; $\epsilon_r \approx 4.6$; multilayer PCB designs are no problem; material loss is significant with $\tan\delta = 0.0180$ (loss tangent)

- **Microfiber PTFE**: Is Polytetrafluoroethylene or Teflon®, with microfibers for reinforcing; Rodgers[3] RT_Duroid 5880 material has $\epsilon_r = 2.20$ (for design) and is very low loss with $\tan\delta = 0.0004$

- **Ceramic-filled PTFE**: Uses ceramic material to increase ϵ_r up to a range of 3.0 to 10.2; Rogers RO3010[4] material has $\epsilon_r = 11.2$ (for design) and $\tan\delta = 0.0022$

- **Alumina**[5]: Aluminum oxide (Al_2O_4) in 99.5% concentration is used for thin-film microwave circuits; $\epsilon_r = 9.9$ and very low loss $\tan\delta = 0.0001$

- **Silicon**: Used for monolithic microwave integrated circuits (MMICs); $\epsilon_r = 11.9$ and very lossy

- **Gallium Arsenide**[6]: Used for monolithic microwave integrated circuits (MMICs); $\epsilon_r = 12.88$ and very low loss $\tan\delta = 0.0004$

- **Sapphire**: Crystalline alumina; $\epsilon_r = 9.0$ (text) and $\tan\delta < 0.0015$

[3] http://www.rogerscorp.com/documents/606/acs/RT-duroid-5870-5880-Data-Sheet.pdf

[4] https://www.rogerscorp.com/documents/722/acs/RO3000-Laminate-Data-Sheet-RO3003-RO3006-RO3010-RO3035.pdf

[5] http://www.microwaves101.com/encyclopedias/alumina-99-5

[6] http://www.microwaves101.com/encyclopedias/gallium-arsenide

Example 2.4: 50 Ω Mstrip on FR-4

- Design a 50 Ω microstrip on FR-4 material having thickness of 1/16 inch. Assume $\epsilon_r = 4.6$. Find the strip width in mils, ϵ_{eff}, and the length a $\lambda/4$ line operating at 2.4 GHz

- The equations in the book can be used straight away, but here we will use the Python functions that are in Chapter 2 IPYNB

- To make things easy first write a function that calculates additional tline parameters

```python
def mstrip_extra(Z0,er,f = 1e9,h = 1):
    """
    Extra parameters: L', C', beta, \lambda_g

    Z0 = characteristic impedance in Ohms
    er = relative permittivity
    f = operatng frequency in Hz
    h = substrate height in m

    Mark Wickert February 2016
    """
    c = 3e8 # m/s
    s, e_eff = mstrip_dsgn(Z0,er)
    C_p = sqrt(e_eff)/(Z0*c)  # F/m
    L_p = Z0**2*C_p   #H/m
    R_p = 0
    G_p = 0
    alpha = 0
    beta = 2*pi/c*sqrt(e_eff)  # rad/m
    lambda_g = c/sqrt(e_eff)/f  # m
    # Print formatted results
    print('Line width w = %1.3e mm, Rel. perm: e_eff = %2.3f' % (s*h*1000,e_eff))
    print('Line width w = %1.3e mils (common PCB unit)' % (s*h*1000/0.0254,))
    print('-----------------------------------------------------')
    print("Resistance per unit length: R' = 0 ohms/m")
    print("Inductance per unit length: L' = %1.3e nH/m" % (L_p*1e9,))
    print("Conductance per unit length: G' = 0 S/n")
    print("Capacitance per unit length: C' = %1.3e pF/m" % (C_p*1e12,))
    print('-----------------------------------------------------')
    print('Phase constant: beta = %1.3e rad/m' % beta)
    print('Wavelength in free space: lambda = %1.3e cm' % (c/f*1e2,))
    print('Wavelength in medium (guide): lambda_g = %1.3e cm' % (lambda_g*1e2,))
```

Figure 2.13: The `mstrip_extra` function.

- A mil is a thousandth of an inch and most PCB shops use mils
- Here $h = 1/16 \cdot 1000 = 62.5$ mils (also 2.54 cm/in)
- The answer is worked out below in the IPYNB

```
mstrip_extra(50,4.6,f= 2.4e9,h=1/16*2.54*1/100) #in*cm/in*m/cm
```

```
Line width w = 2.931e+00 mm, Rel. perm: e_eff = 3.460
Line width w = 1.154e+02 mils (common PCB unit)
-----------------------------------------------------
Resistance per unit length: R' = 0 ohms/m
Inductance per unit length: L' = 3.100e+02 nH/m
Conductance per unit length: G' = 0 S/n
Capacitance per unit length: C' = 1.240e+02 pF/m
-----------------------------------------------------
Phase constant: beta = 3.896e-08 rad/m
Wavelength in free space: lambda = 1.250e+01 cm
Wavelength in medium (guide): lambda_g = 6.720e+00 cm
```

```
print('Quarterwave line length at 2.4 GHz  = %1.3f cm' % (6.72/4,))
print('or %1.3f in.' % (6.72/4*1/2.54,))
```

```
Quarterwave line length at 2.4 GHz  = 1.680 cm
or 0.661 in.
```

Figure 2.14: The desired calculations plus some extras.

- In summary, the line width $W = 115.4$ mils (2.931 mm), $\epsilon_{\text{eff}} = 3.460$, and $\lambda_g/4 = 661$ mils (1.680 cm)

Example 2.5: Design/Analysis Charts for PCB Mstrip

- Produce some useful design/analysis charts for use on quizzes and exams
- The relative permittivity values are taken from plastic substrates used in PCBs

- We obtain plots of Z_0 versus $s = w/h$ and ϵ_{eff} versus w/h using the Python functions described earlier.

```
s = arange(0.01,10,.01)
Z0 = zeros_like(s)
e_eff = zeros_like(s)
er = (2.2,4.6,11.2)
for m in range(3):
    for k,sk in enumerate(s):
        Z0[k],e_eff[k] = mstrip_anal(sk,er[m])
    plot(s,e_eff)
ylim([0,10])
xlabel(r'$w/h$')
ylabel(r'$\epsilon_{eff}$')
title(r'Microstrip Analysis: $\epsilon_{eff}$ vs $w/h$')
legend((r'MF-PTFE: $\epsilon_r = 2.2$',r'FR-4: $\epsilon_r=4.6$',
        r'CF_PTFE: $\epsilon_r=11.2$'),loc='best')
grid();
```

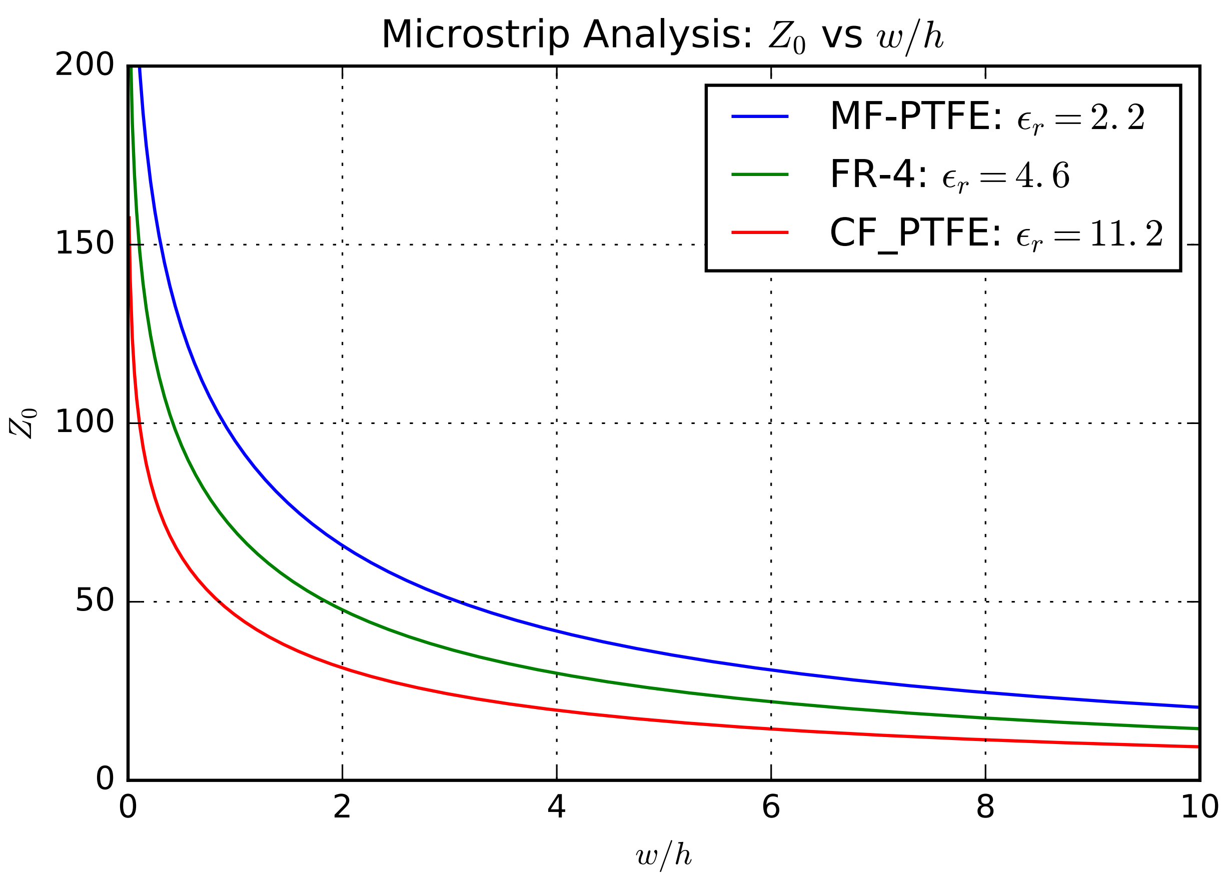


Figure 2.15: Python commands followed by Z_0 versus w/h plot.

```
s = arange(0.01,10,.01)
Z0 = zeros_like(s)
e_eff = zeros_like(s)
er = (2.2,4.6,11.2)
for m in range(3):
    for k,sk in enumerate(s):
        Z0[k],e_eff[k] = mstrip_anal(sk,er[m])
    plot(s,e_eff)
ylim([0,10])
xlabel(r'$w/h$')
ylabel(r'$\epsilon_{eff}$')
title(r'Microstrip Analysis: $\epsilon_{eff}$ vs $w/h$')
legend((r'$\epsilon_r = 2.2$',r'$\epsilon_r=4.6$',
        r'$\epsilon_r=11.2$'),loc='best')
grid();
```

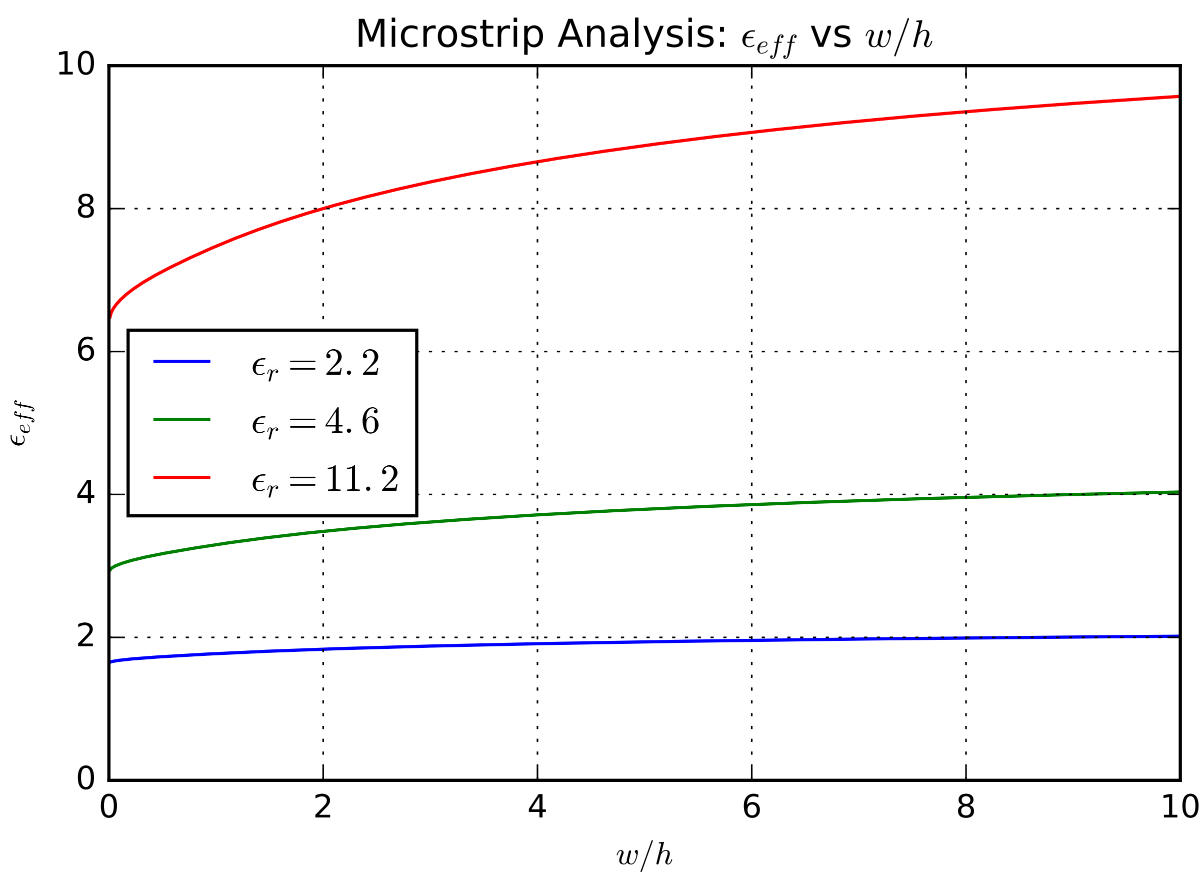


Figure 2.16: Python commands followed by ϵ_{eff} versus w/h plot.

- Larger versions of the plots themselves are available for download (soon)

2.6 The Lossless Transmission Line: General Considerations

- Up to this point we have seen that Z_0 and γ play a central role in the characterization of a transmission line
- Modeling with a low-loss dielectric and high conductivity conductors is a good place to start a design
- Under the assumption that $R' = G' = 0$ we have

$$\gamma = \alpha + j\beta = j\omega\sqrt{L'C'}$$

so

$$\boxed{\begin{aligned} \alpha &= 0 && \text{(Lossless)} \\ \beta &= \omega\sqrt{L'C'} && \text{(Lossless)} \end{aligned}}$$

- The characteristic impedance becomes

$$\boxed{Z_0 = \sqrt{\frac{L'}{C'}} \quad \text{(Lossless)}}$$

- Summarizing some other useful results into one location

$$\boxed{\begin{aligned} \beta &= \omega\sqrt{\mu\epsilon} \quad \text{(rad/m)} \\ u_p &= \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\epsilon_r}} \quad \text{(m/s)} \\ \lambda &\overset{\text{also}}{=} \lambda_g = \frac{u_p}{f} = \frac{c}{f} \cdot \frac{1}{\sqrt{\epsilon_r}} = \frac{\lambda_0}{\sqrt{\epsilon_r}} \quad \text{(m)} \end{aligned}}$$

Note: The notation λ_g refers to wavelength in a *guided* medium, most often a waveguide or a TEM transmission line; $\lambda_0 = c/f$ is of course the free space wavelength

- The table below summarizes lossless line parameters, γ, $u_p = v_p$, and Z_0, for the four classical tline structures originally shown in Figure 2.3(a)–(d).

Table 2.3: Summary of lossless line parameters for the four classical line types.

	Propagation Constant $\gamma = \alpha + j\beta$	**Phase Velocity** u_p	**Characteristic Impedance** Z_0
General case	$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$	$u_p = \omega/\beta$	$Z_0 = \sqrt{\dfrac{(R' + j\omega L')}{(G' + j\omega C')}}$
Lossless $(R' = G' = 0)$	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = \sqrt{L'/C'}$
Lossless coaxial	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = (60/\sqrt{\varepsilon_r})\ln(b/a)$
Lossless two-wire	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = (120/\sqrt{\varepsilon_r}) \cdot \ln[(D/d) + \sqrt{(D/d)^2 - 1}]$ $Z_0 \approx (120/\sqrt{\varepsilon_r})\ln(2D/d)$, if $D \gg d$
Lossless parallel-plate	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = (120\pi/\sqrt{\varepsilon_r})\,(h/w)$

Notes: (1) $\mu = \mu_0$, $\varepsilon = \varepsilon_r\varepsilon_0$, $c = 1/\sqrt{\mu_0\varepsilon_0}$, and $\sqrt{\mu_0/\varepsilon_0} \approx (120\pi)\ \Omega$, where ε_r is the relative permittivity of insulating material. (2) For coaxial line, a and b are radii of inner and outer conductors. (3) For two-wire line, d = wire diameter and D = separation between wire centers. (4) For parallel-plate line, w = width of plate and h = separation between the plates.

Example 2.6: Solve for ϵ_r Given f and λ_g

- The wavelength in a lossless transmission line is known to be 5.828 cm at a frequency of 2.4 GHz
- Find ϵ_r of the tline insulating material
- We know that $\lambda_g = \lambda_0/\sqrt{\epsilon_r}$ and $\lambda_0 = c/f$, so putting these

two equations together we have

$$\epsilon_r = \left(\frac{\lambda_0}{\lambda_g}\right)^2 = \left(\frac{c/f}{\lambda_g}\right)^2$$
$$= \left(\frac{3 \times 10^8/2.4 \times 10^9}{0.05828}\right)^2 = 4.60$$

Example 2.7: Line Parameters from ϵ_r and L'

- Given a lossless line has $\epsilon_r = 6$ and the line inductance is $L' = 0.8\ \mu$H/m
- Find u_p, C' and Z_0
- To find u_p we use the fact that

$$u_p = \frac{c}{\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{\sqrt{6}} = 1.225 \times 10^8 \quad \text{(m/s)}$$

- To find C' we use the fact that $\beta = \omega\sqrt{L'C'} \stackrel{\text{also}}{=} \omega\sqrt{\mu\epsilon}$, so

$$C' = \frac{\mu\epsilon_0\epsilon_r}{L'} = \frac{4\pi \times 10^{-7} \cdot 8.85 \times 10^{-12} \cdot 6}{0.8 \times 10^{-6]}} = 83.42 \quad \text{(pF/m)}$$

- Finally, finding Z_0 makes use of

$$Z_0 = \frac{L'}{C'} = \frac{0.8 \times 10^{-6}}{83.42 \times 10^{-12}} = 97.94 \quad (\Omega)$$

2.6.1 Voltage Reflection Coefficient

- In this subsection we impose the boundary conditions of the following circuit

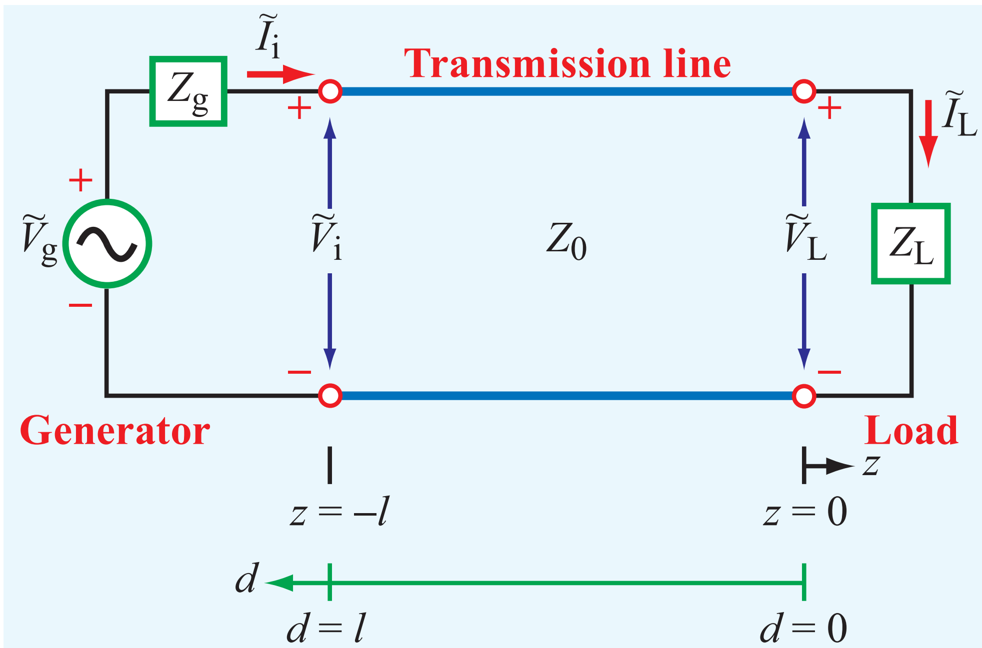


Figure 2.17: Boundary conditions imposed by placing a lossless tline between a generator and load.

- The total voltage and current, under the lossless assumption, makes

$$\tilde{V}(z) = V_0^+ e^{-\beta z} + V_0^- e^{\beta z}$$
$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\beta z} - \frac{V_0^-}{Z_0} e^{\beta z}$$

and in particular at the load end of the line, where $z = 0$, we have

$$\tilde{V}_L \stackrel{\text{also}}{=} \tilde{V}(z = 0) = V_0^+ + V_0^-$$
$$\tilde{I}_L \stackrel{\text{also}}{=} \tilde{I}(z = 0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

- From Ohms law for impedances it is also true that

$$\tilde{Z}_L \stackrel{\text{must}}{=} \frac{\tilde{V}_L}{\tilde{I}_L},$$

so rearranging to solve for V_0^- in terms of V_0^+ results in

$$V_0^- = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_0^+$$

which establishes a relationship between the incident voltage wave amplitude V_0^+ and the reflected voltage wave amplitude V_0^-

- **Definition**: The *voltage reflection coefficient* Γ, is

$$\boxed{\Gamma = \frac{V_0^-}{V_0^+} \equiv \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) = \frac{z_L - 1}{z_L + 1},}$$

where $z_L = Z_L/Z_0$ is the *normalized* load impedance

- A similar relationship holds for I_0^-/I_0^+, except due to $V_0^{=}/I_0^+ = -V_0^-/I_0^+$, we have

$$\boxed{\frac{I_0^-}{I_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma}$$

- It is also worth noting that since Z_L is in general complex, Γ is complex with polar form $|\Gamma|e^{j\theta_r}$
  - For a passive load Z_L it turns out that $|\Gamma| \leq 1$

Matched Load

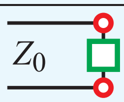
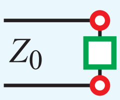
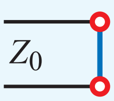
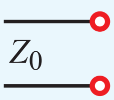
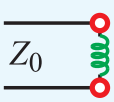
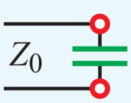
When

$$V_0^- = 0 \;\Rightarrow\; \Gamma_L = 0 \;\Rightarrow\; Z_L = Z_0$$

making the load *matched* to the tline characteristic impedance.

Reflection Coefficient Special Cases

Table 2.4: Reflection coefficient under special cases ($r = R/Z_0$ and $jx = jX/Z_0$).

	Reflection Coefficient $\Gamma = \lvert\Gamma\rvert e^{j\theta_r}$	
Load	$\lvert\Gamma\rvert$	θ_r
 $Z_L = (r + jx)Z_0$	$\left[\dfrac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2}$	$\tan^{-1}\left(\dfrac{x}{r-1}\right) - \tan^{-1}\left(\dfrac{x}{r+1}\right)$
 Z_0	0 (no reflection)	irrelevant
 (short)	1	$\pm 180^\circ$ (phase opposition)
 (open)	1	0 (in-phase)
 $jX = j\omega L$	1	$\pm 180^\circ - 2\tan^{-1} x$
 $jX = \dfrac{-j}{\omega C}$	1	$\pm 180^\circ + 2\tan^{-1} x$

Example 2.8: **Series RC Load**

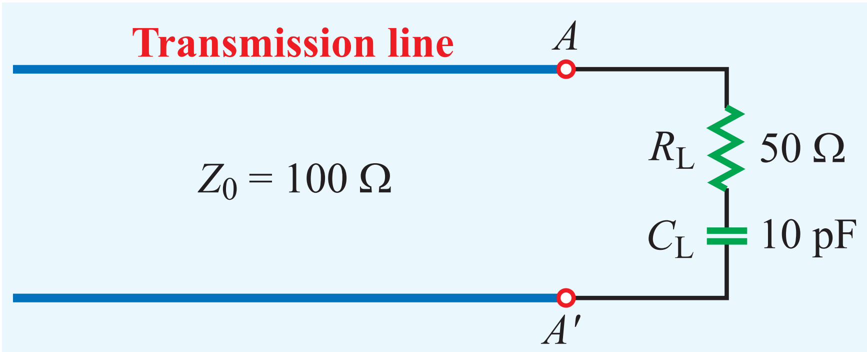


Figure 2.18: RC load on a $Z_0 = 100\ \Omega$ lossless tline.

- Consider a $Z_0 = 100\ \Omega$ lossless tline driving a load Z_L composed of resistor $R_L = 50\ \Omega$ in series with capacitor $C_L = 10$ pF
- Find Γ_L and check the result using LTspice $Gamma_L$
- Here we use Python, MATLAB or Excel would also work

```
f = 100e6
Z0 = 100
RL = 50
CL = 10e-12
Z_CL = 1/(1j*2*pi*f*CL)
ZL = RL + Z_CL
GammaL = (ZL - Z0)/(ZL + Z0)
print('GammaL ' + cpx_fmt(GammaL,'polar'))
```

```
GammaL 0.7628 /_-60.7444 (deg)
```

Figure 2.19: Python calcuation of Γ_L.

- LTspice has both loss and lossy transmission line models
- To make LTspice work in this problem we use an ideal tlie having $Z_0 = 100\ \Omega$, arbitary line length specified by a time delay $T_d = l/u_p$, and source impedance $Z_g = Z_0$ LTspice can only

measure voltage and currents at circuit nodes and branches, respectively

- We desire V_o^+, V_0^-, and Γ_L

- From the original boundary conditions at the load,

$$\begin{aligned} \tilde{V}_L &= V_0^+ + V_0^- \\ \tilde{I}_L &= \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} \\ \text{or} \quad \tilde{I}_L Z_0 &= V_0^+ - V_0^- \end{aligned}$$

- By adding and subtracting the first and last equations above, we obtain

$$\boxed{\begin{aligned} V_0^+ &= \frac{\tilde{V}_L + \tilde{I}_L Z_0}{2} \\ V_0^- &= \frac{\tilde{V}_L - \tilde{I}_L Z_0}{2} \end{aligned}}$$

and also the relationship

$$\boxed{\Gamma_L = \frac{V_0^-}{V_0^+} = \frac{\tilde{V}_L - \tilde{I}_L Z_0}{\tilde{V}_L + \tilde{I}_L Z_0}}$$

- In this example we use the Γ_L equation above to post-process the spice voltage and current at the load as shown in the figures below

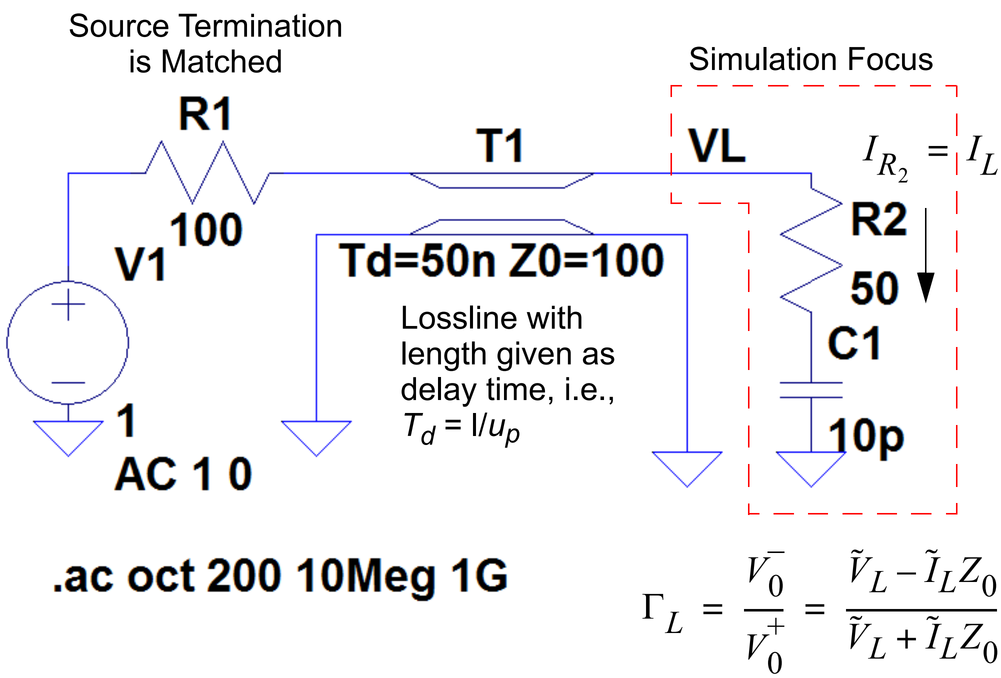


Figure 2.20: LTspice model emplying a lossless transmission driven by a source having matching impedance of $Z_g = Z_0 = 100\ \Omega$.

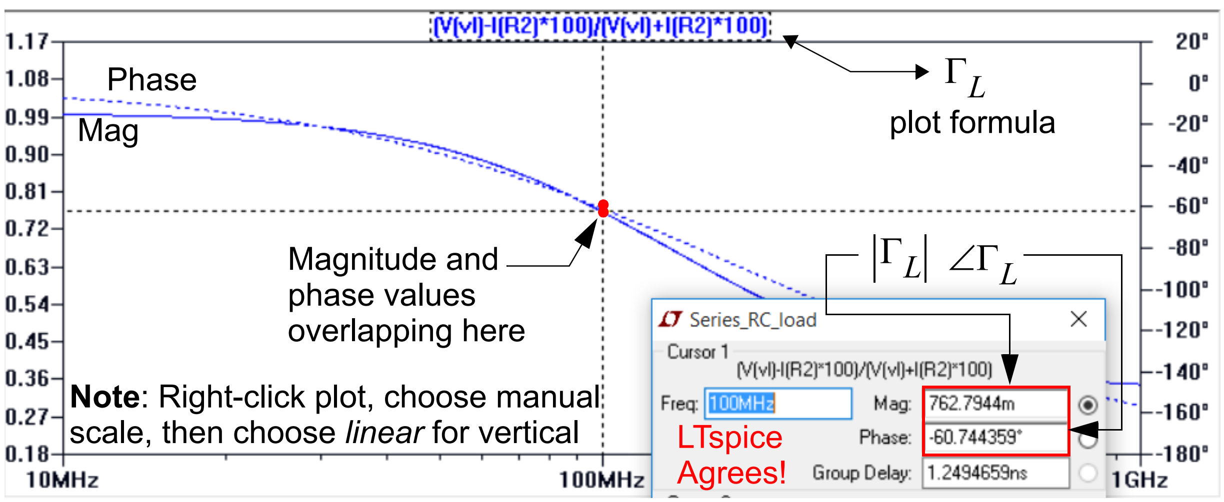


Figure 2.21: LTspice AC steady-state results with total voltage and current converted to the voltage reflection coefficient (use plot linear mode).

- The results from both calculation means agree!

Example 2.9: **Series RC Load**

- Consider the case of a purely reactive load $Z_L = jX_L$
- Calculating the reflection coefficient we have

$$\begin{aligned}\Gamma_L &= \frac{jX_L - Z_0}{jX_L + Z_0} = \frac{-(Z_0 - jX_L)}{Z_0 + jX_L}\\ &= -1e^{-j2\theta},\end{aligned}$$

where $\theta = \angle(Z_0 + jX_L)$

- The significant result is that $|\Gamma_L| = 1$ independent of the value of X_L
- The angle of the angle θ does change with changes in X_l relative to Z_0

2.6.2 Standing Waves

- Knowing now that $V_0^- = \Gamma V_0^=$ means that we can write

$$\begin{aligned}\tilde{V}(z) &= V_0^+\big(e^{-j\beta z} + \Gamma e^{j\beta z}\big)\\ \tilde{V}(z) &= \frac{V_0^+}{Z_0}\big(e^{-j\beta z} - \Gamma e^{j\beta z}\big)\end{aligned}$$

- As we move along the line for $-l \le z \le 0$, a curiosity is what is the nature of $|\tilde{V}(z)|$ and $\tilde{I}(z)$

- To answer that we do some math analysis, such as

$$\begin{aligned}|\tilde{V}(z)| &= \sqrt{\tilde{V}(z)\cdot\tilde{V}^*(z)} \\ &= |V_0^+|\left[\left(e^{-j\beta z}+\Gamma e^{-j\beta z}\right)\left(e^{j\beta z}+\Gamma^* e^{j\beta z}\right)\right]^{1/2} \\ &= |V_0^+|\left[1+|\Gamma|^2+2|\Gamma|\cos(2\beta x-\theta_r)\right]^{1/2},\end{aligned}$$

  where in polar form $\Gamma = |\Gamma|e^{j\theta_r}$

- A similar analysis for $\tilde{I}(z)$ can be performed with $|V_0^+| \to |V_0^+|/Z_0$ and flipping the sign of the cosine term

- It is convenient to view the voltage and current magnitude in terms of a positive distance d back from the load, so we let $d = -z$

- Finally, we have

$$\begin{aligned}|\tilde{V}(d)| &= |V_0^+|\left[1+|\Gamma|^2+2|\Gamma|\cos\left(2\beta d-\theta_r\right)\right]^{1/2} \\ |\tilde{I}(d)| &= \frac{|V_0^+|}{Z_0}\left[1+|\Gamma|^2-2|\Gamma|\cos\left(2\beta d-\theta_r\right)\right]^{1/2}\end{aligned}$$

- These two expressions generate what is known as the *standing wave* pattern on the line

- Using a *slotted-line*, you can physically measure the voltage amplitude (magnitude) along the line by sliding a carriage along a track that has a probe inserted into a slit cut into an air-line

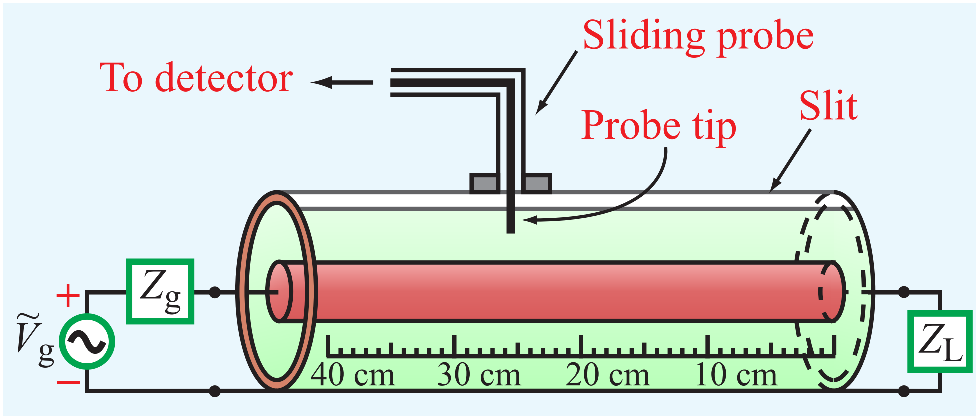


Figure 2.22: Slotted coaxial line with E-field probe to measure the standing wave magnitude.

- The expressions just derived are implemented in the Chapter 2 IPYNB

```
d2lam  = arange(0,1,0.005) # d/lambda for plotting
V0_p_mag = 1.0 # V
Z0 = 50 # Ohms
Gamma = 0.8*exp(1j*45*(pi/180))
#Gamma = -1.0
V_abs = V0_p_mag*(1 + abs(Gamma)**2 + \
                  2*abs(Gamma)*cos(2*(2*pi*d2lam) - angle(Gamma)))**0.5
I_abs = V0_p_mag/Z0*(1 + abs(Gamma)**2 - \
                     2*abs(Gamma)*cos(2*(2*pi*d2lam) - angle(Gamma)))**0.5
subplot(211)
plot(-d2lam,V_abs)
ylim([0,2])
xlabel(r' Wavelength Normalized Distance $-d/\lambda$ or $z\lambda$')
ylabel(r'$|\tilde{V}(d)|$ (V)')
title(r'V and I Standing-Wave Patterns: $\Gamma$ = ' + \
      cpx_fmt(Gamma,'polar',d2=2))
grid()
subplot(212)
plot(-d2lam,I_abs*1e3)
ylim([0,2/Z0*1e3])
xlabel(r' Wavelength Normalized Distance $-d/\lambda$ or $z/\lambda$')
ylabel(r'$|\tilde{I}(d)|$ (mA)')
grid()
tight_layout()
```

Figure 2.23: Python code for plotting standing wave patterns.

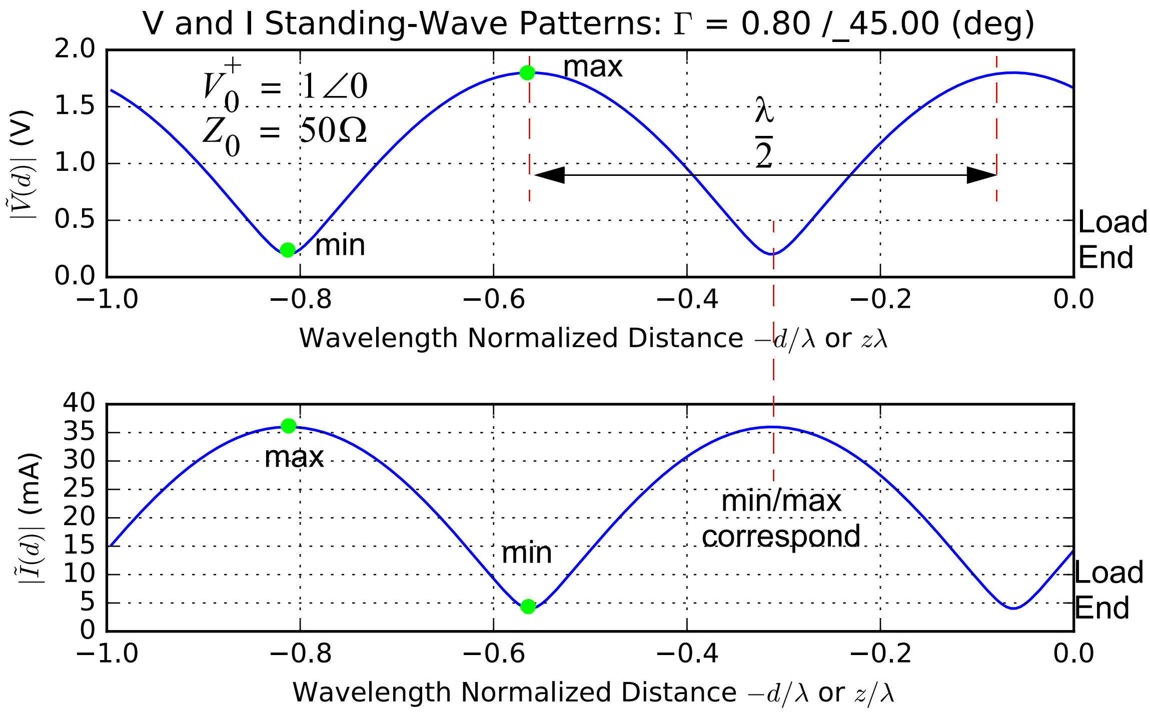


Figure 2.24: Plots of $|\tilde{V}(d)|$ and $\tilde{I}(z)|$ for $\Gamma = 0.8\angle 45°$.

- The 2β term is responsible for the period of the pattern being $\lambda/2$

- Clearly the voltage maximum (constructive interference) value corresponds to $V_0^+(1 + |\Gamma|)$

- Similarly, voltage minimum (destructive interference) value corresponds to $V_0^+(1 - |\Gamma|)$

- **Special Cases:**

  - Matched $\Rightarrow \Gamma = 0$, is a flat line at $|V_0^+|$
  - Short $\Rightarrow \Gamma = -1$, begins with $|\tilde{V}(0)| = 0$ an d the maximum values going to $2|V_0^+|$ and minimums going to zero (looks like a half-wave rectified sine wave)

– Open $\Rightarrow$ $\Gamma = 1$, begins with $|\tilde{V}(0)| = 2|V_0^+|$ and minimum values going to zero (phased 90° relative the short case)

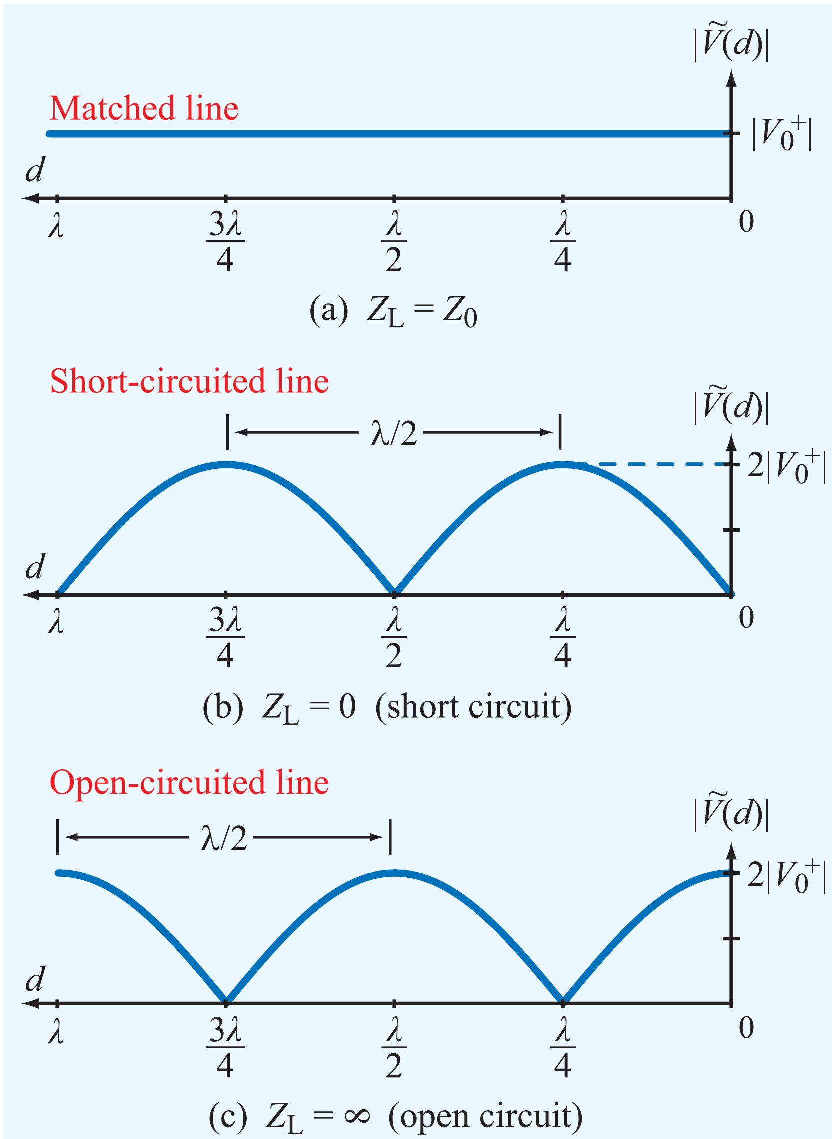


Figure 2.25: Standing-waves for matched, short, and open cases.

Maximum and Minimum Values

- The maximum value of the voltage magnitude, $|\tilde{V}|_{\max} = |V_0^+|[1+|\Gamma|]$ occurs when the argument of the cos() term is an even multiple of π i.e.,

$$2\beta d_{\max} - \theta_r = 0,\ 2\pi, \ldots$$

- Similarly the voltage magnitude minimum, $|\tilde{V}|_{\min} = |V_0^+|[1 - |\Gamma|]$, occurs when the argument of the cos() term is an odd multiple of π i.e.,

$$2\beta d_{\min} - \theta_r = \pi, 3\pi, \ldots$$

- The maximum value of the current magnitude, $|\tilde{I}|_{\max} = |I_0^+|[1+|\Gamma|]$ occurs when the argument of the cos() term is an odd multiple of π i.e.,

$$2\beta d_{\max} - \theta_r = \pi, 3\pi, \ldots$$

- The maximum value of the current magnitude, $|\tilde{I}|_{\min} = |I_0^+|[1-|\Gamma|]$ occurs when the argument of the cos() term is an even multiple of π i.e.,

$$2\beta d_{\min} - \theta_r = 0,\ 2\pi, \ldots$$

- In all cases minimum and maximum values are separated by $\lambda/4$ and voltage maxim correspond to current minimum, etc.

2.6.3 VSWR

- The ratio of $|\tilde{V}|_{\max}$ to $|\tilde{V}|_{\max}$ is given the name *voltage standing-wave ratio* (VSWR) or (SWR) and in the text is denoted by S

$$\boxed{\begin{aligned} S &= \frac{|\tilde{V}|_{\max}}{|\tilde{V}|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \\ |\Gamma| &\stackrel{\text{also}}{=} \frac{S-1}{S+1} \end{aligned}}$$

- VSWR is an indicator of the mismatch between Z_0 and Z_L
- **Note**: $\Gamma = 1 \Rightarrow S = 1$ and $|\Gamma| = 1 \Rightarrow S = \infty$

Example 2.10: $|\Gamma|$ from VSWR

- Given **VSWR** $= S = 5$ find $|\Gamma|$

$$|\Gamma| = \frac{5-1}{5+1} = 0.667$$

Example 2.11: Find L in Parallel RL Giving $S = 5$

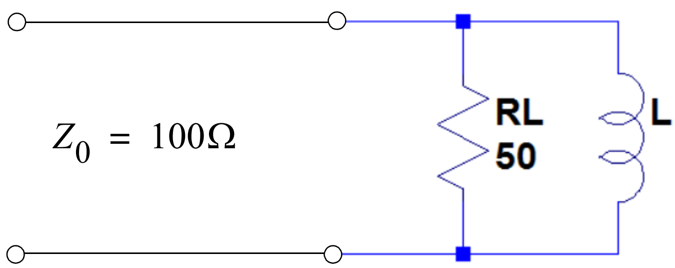


Figure 2.26: An RL Load Terminating a $Z_0 = 100$ tline.

- Consider the circuit shown above with L unknown, $S = \text{VSWR} = 5$, and $f = 100$ MHz
- To start with we know that

$$S \equiv \frac{1+|\Gamma|}{1-|\Gamma|} \Rightarrow |\Gamma| = \frac{S-1}{S+1},$$

so

$$|\Gamma| = \frac{5-1}{5+1} = \frac{2}{3}$$

- Now to solve for Γ in terms of L we first find Z_L to be

$$Z_L = \frac{R_L \cdot j\omega L}{R_L + j\omega L}, \quad \omega = 2\pi f$$

next plugging into the expression for Γ

$$\begin{aligned}\Gamma &= \frac{\frac{R_L \cdot j\omega L}{R_L + j\omega L} - Z_0}{\frac{R_L \cdot j\omega L}{R_L + j\omega L} + Z_0} \\ &= \frac{j\omega R_L L - (Z_0 R_L) - j\omega Z_0 L}{j\omega R_L L + (Z_0 R_L) - j\omega Z_0 L} \\ &= \frac{-Z_0 R_L - j\omega(Z_0 - R_L)L}{Z_0 R_L - j\omega(Z_0 + R_L)L}\end{aligned}$$

- To isolate L find $|\Gamma|^2$ and then rearrange

$$|\Gamma|^2 = \frac{(Z_0 R_L)^2 + \omega^2 (Z_0 - R_L)^2 L^2}{(Z_0 R_L)^2 + \omega^2 (Z_0 + R_L)^2 L^2}$$

so working the algebra

$$\boxed{L = \sqrt{\frac{(Z_0 R_L)^2 (1 - |\Gamma|^2)}{\omega^2 \big((Z_0 + R_L)^2 |\Gamma|^2 - (Z_0 - R_L)^2\big)}}}$$

- Using Python for the calculations (also in Chapter IPYNB) we arrive at

$$L = 68.489 \quad \text{(nH)}$$

```
Gam_mag = 2/3
Z0 = 100
RL = 50
w = 2*pi*100e6
L = sqrt((Z0*RL)**2*(1-(Gam_mag)**2)/(w**2*((Z0+RL)**2*Gam_mag**2-(Z0-RL)**2)))
print('L = %1.3f (nH)' % (L*1e9,))
```

```
L = 68.489 (nH)
```

Figure 2.27: Calculation of L.

- A less elegant approach is to numerically find L by plotting $|\Gamma|$ versus L and then numerically find the L value which makes $|\Gamma| = 2/3$

```
# Plot Abs[Gamma] versus C
L = arange(1,200)*1e-9
Gamma_mag = abs((-5000 - 1j*2*pi*100e6*L*50)/(5000 + 1j*2*pi*100e6*L*150))
plot(L*1e9,Gamma_mag)
title(r'Searching for $L$ Numerically by Plotting $|\Gamma|$ vs $L$')
ylabel(r'$|\Gamma|$')
xlabel(r'$L$ in nH')
grid();
```

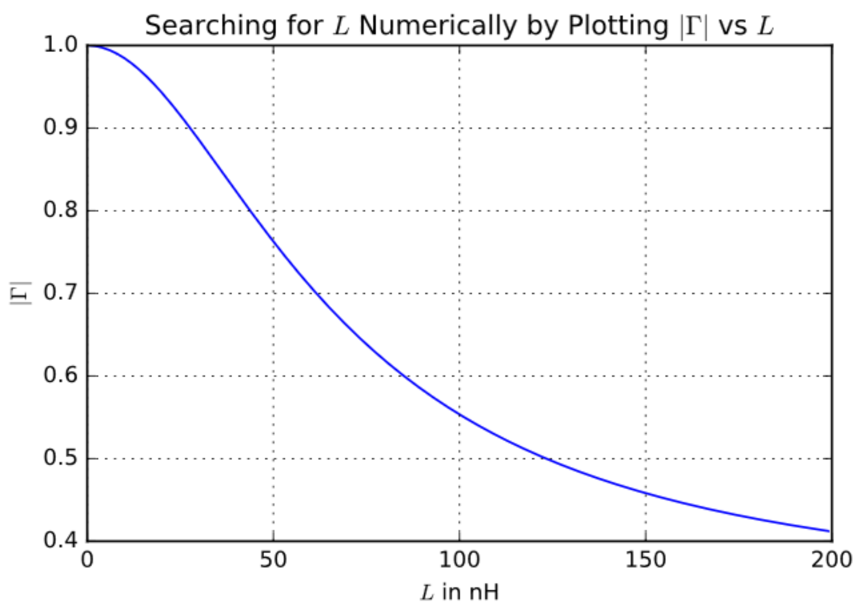


Figure 2.28: Plotting of $|\Gamma|$ versus L.

- A precise numerical solution can be found by using a solver such as `fsolve`, which is available in the `scipy.optimize` package

- We start by forming an *objective* function that will instead cross through zero at the desired L value, that is search for L such that $F_{\text{obj}}(L) = |\Gamma(L)| - 2/3 = 0$

```
from scipy.optimize import fsolve
```

```
def Gamma_abs_objective(L):
    """
    |Gamma| - 2/3 as a function of L

    Mark Wickert February 2016
    """
    L = L*1e-9
    RL = 50
    Z0 = 100
    f = 100e6
    w = 2*pi*f
    Gamma_abs = abs(((RL*1j*w*L)/(RL+1j*w*L)-Z0)/((RL*1j*w*L)/(RL+1j*w*L)+Z0))
    return Gamma_abs - 2/3 # Looking for zero crossing
```

```
L_req = fsolve(Gamma_abs_objective,100,xtol = 1e-5)
print('L_req = %1.1f nH' % (L_req,))
```

```
L_req = 68.5 nH
```

Figure 2.29: Using `fslove` to numerically search for L.

- The results is the same in both cases
- Just pulling the L value from the plot is easier still, and quite accurate when you zoom in

2.7 Wave Impedance of a Lossless Line

- Anywhere along the line we can form the ratio $\tilde{V}(d)/\tilde{I}(d)$, hence this voltage to current ratio is given the name *wave impedance*

- Drilling into this definition reveals

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+\left[e^{j\beta d} + \Gamma e^{-j\beta d}\right]}{V_0^+\left[e^{j\beta d} - \Gamma e^{-j\beta d}\right]} Z_0$$
$$= Z_0\left[\frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}}\right] \quad (\Omega)$$

- If we define the reflection coefficient at location $d = -z$ as

$$\boxed{\Gamma(d) \equiv \Gamma e^{-j2\beta d}}$$

  then we can also write

$$Z(d) = Z_0\left[\frac{1 + \Gamma_d}{1 - \Gamma_d}\right] \quad (\Omega)$$

- **To be clear**:
  - $Z(d)$ is the ratio of the total voltage to the total current (both incident and reflected)
  - Z_0 is the ratio of just the incident voltage over the incident current (minus sign on reflected voltage over reflected current)

- A great use of the wave impedance concept is in equivalent circuits as shown below

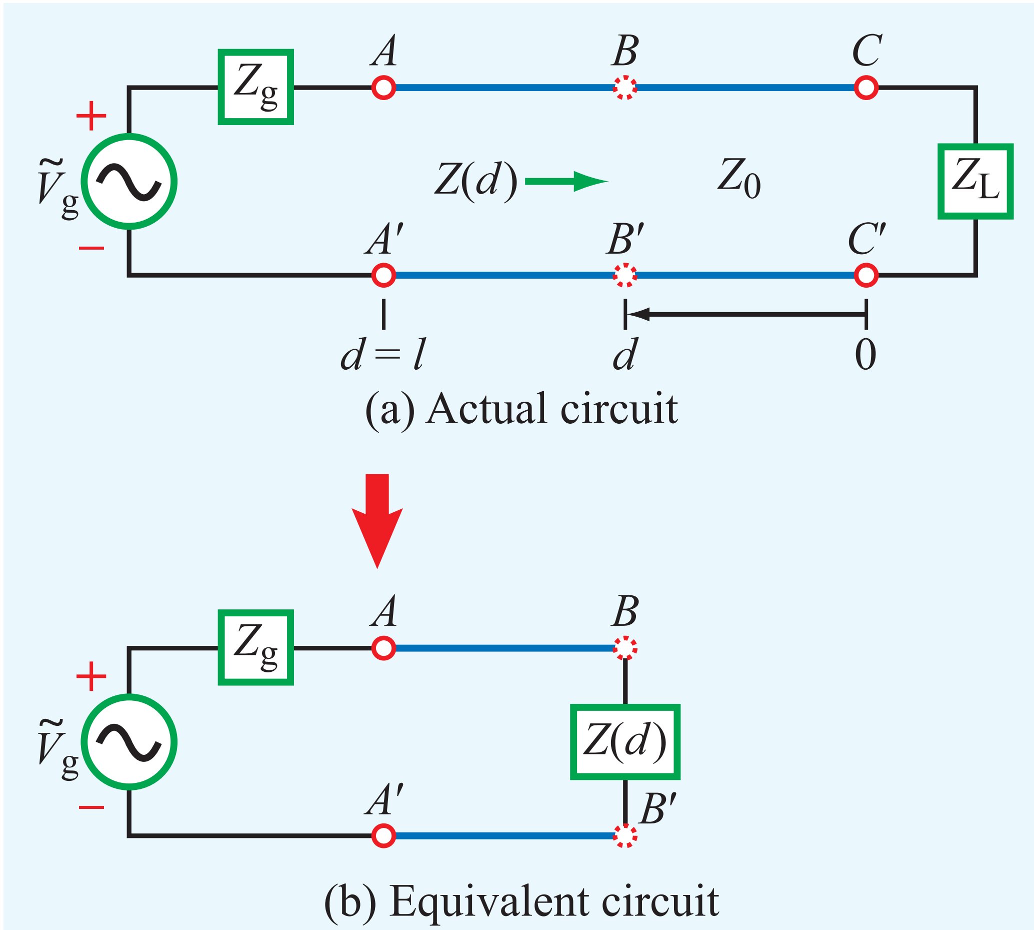


Figure 2.30: Significance of the wave impedance in tline circuit modeling.

Line Input Impedance

- As a special case consider the line input impedance

$$Z_{in} = Z(l = d) = Z_0 \left[\frac{1 + \Gamma_l}{1 - \Gamma_l} \right]$$

- Noting that

$$\Gamma_l = \Gamma e^{-j2\beta l}$$

allows us to write

$$\begin{aligned} Z_{\mathbf{in}} &= Z_0\left(\frac{Z_L\cos(\beta l)+jZ_0\sin(\beta l)}{Z_0\cos(\beta l)+jZ_L\sin(\beta l)}\right) \\ &= Z_0\left(\frac{Z_L+jZ_0\tan(\beta l)}{Z_0+jZ_L\tan(\beta l)}\right) \end{aligned}$$

also making use of Euler's identity and then dividing top and bottom by cos()

- **Application**: Find the voltage at the source end of the line $\tilde{V}_i = \tilde{V}(z = -l)$ (remember here the notation assumes z) using Z_{in}, then go on to find the incident voltage V_0^+
  - From simple circuit analysis (voltage divider)

$$\tilde{V}_i = \tilde{I}_i Z_{\text{in}} = \frac{\tilde{V}_g Z_{\text{in}}}{Z_g + Z_{\text{in}}}$$

  - It is also true that

$$\tilde{V}_i = \tilde{V}(z=-l) = V_0^+\left[e^{j\beta l} + \Gamma e^{-j\beta l}\right]$$

  - Solving for V_0^+ yields

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{\text{in}}}{Z_g + Z_{\text{in}}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$

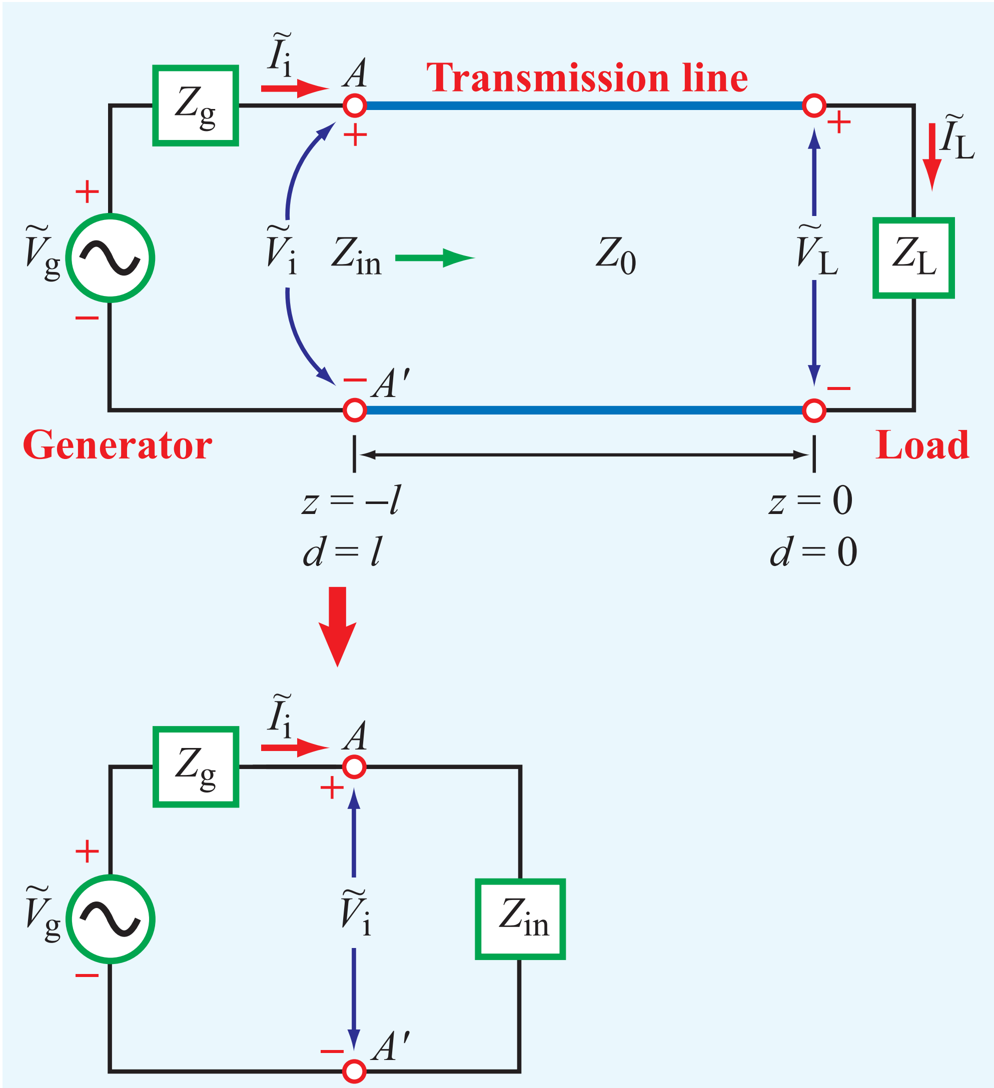


Figure 2.31: How to obtain V_0^+ using Z_{in} to obtain $\tilde{V}_i$ and $\tilde{I}_i$, and then V_0^+ using $\tilde{V}_i$.

- Finally obtaining V_0^+ completes the solution of the 1D transmission line wave equation; **everything is known!**

2.8 Special Cases of the Lossless Line

- In this section we consider the input impedance of a lossless transmission line under the special cases of: (i) a short at the load, (ii) an open at the load, and (iii) a pure resistive load $R_L \neq Z_0$ at the load
- This study opens the door to filter design and impedance matching circuits, and the use of the Smith Chart

2.8.1 Short-Circuited Line

- For a short circuited line $Z_L = 0$ so at distance d away from the load the voltage and current are given by

$$\tilde{V}_{\text{sc}}(d) = V_0^+\big[e^{-j\beta d} + \overbrace{\Gamma}^{-1} e^{-j\beta d}\big] = 2jV_0^+ \sin\beta d$$
$$\tilde{I}_{\text{sc}}(d) = \frac{V_0^+}{Z_0}\big[e^{-j\beta d} - \underbrace{\Gamma}_{-1} e^{-j\beta d}\big] = \frac{2V_0^+}{Z_0}\cos\beta d$$

- The input impedance of a length l line is

$$\boxed{Z_{\text{in}}^{\text{sc}}(l) = Z_{\text{in}}(l)\big|_{Z_L=0} = jZ_0\tan(\beta l)}$$

- The impedance is purely reactive, but changing with d as we move back from the short circuit load (d increasing)
- The figure below shows that the reactance is positive, as in an inductor, part of the time and negative, as in a capacitor, at other times

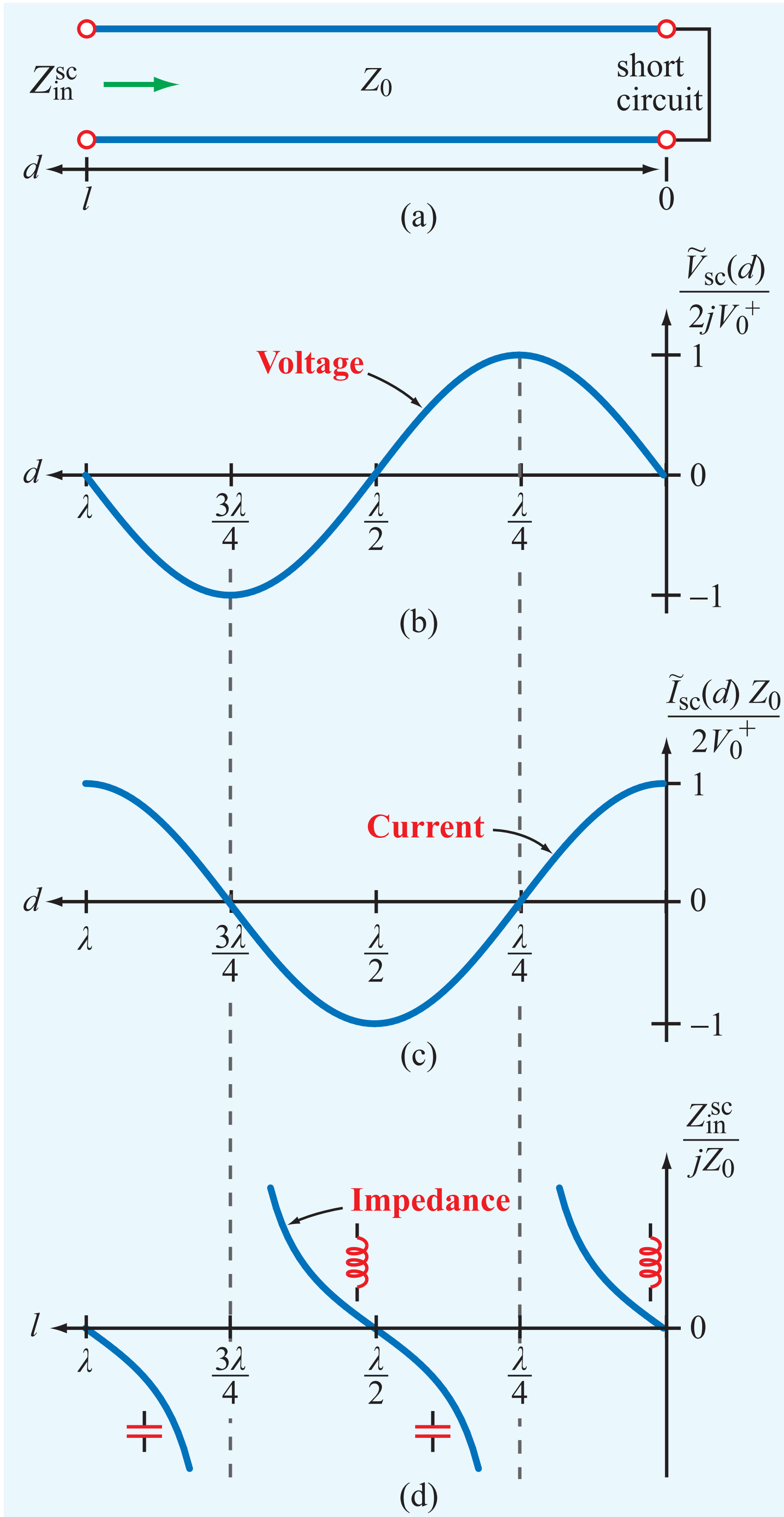


Figure 2.32: Short-circuited line showing voltage, current and $Z_{\mathbf{in}}^{\mathbf{sc}}/jZ_0$ versus l.

- **Inductor Like**: Set

$$j\omega L_{\text{eq}} = jZ_0 \tan(\beta l)$$

or

$$L_{\text{eq}} = \frac{Z_0 \tan(\beta l)}{\omega} \quad \text{(H)}$$

provided $\tan(\beta l) > 0$

- **Capacitor Like**: Set

$$\frac{1}{j\omega C_{\text{eq}}} = jZ_0 \tan(\beta l)$$

or

$$C_{\text{eq}} = -\frac{1}{Z_0 \omega \tan(\beta l)} \quad \text{(F)}$$

provided $\tan(\beta l) < 0$

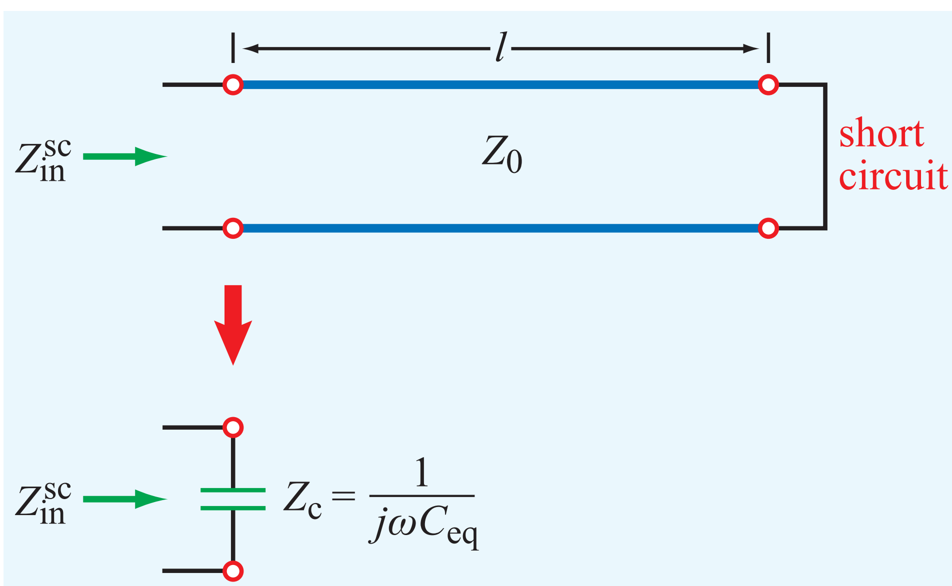


Figure 2.33: Short-circuited line behaves as a capacitor depending upon the line length l.

Impedance versus Frequency

- In microwave circuit design considering the $Z_{\text{in}}^{\text{sc}}$ relationship versus frequency of interest, as circuits typically operate over a band of frequencies

- To take this view we write

$$\beta l \overset{\text{elect. length}}{=} \theta = \frac{\pi}{2} \cdot \frac{f}{f_0}$$

  where θ is line electrical length in radians and f_0 is the corresponding quarter-wave frequency of the line

- The quarter-wave frequency f_0 follows from the fact that $\lambda_0 = u_p/f_0$ or $f_0 = u_p/\lambda_0$

- Now,

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda_0}{4} = \frac{2\pi}{u_p/f} \cdot \frac{u_p}{4f_0} = \frac{pi}{2} \cdot \frac{f}{f_0}$$

- Expressing $Z_{\text{in}}^{\text{sc}}$ as function frequency, we have

$$Z_{\text{in}}^{\text{sc}}(f) = jZ_0 \tan(\theta) = jZ_0 \tan\left(\frac{\pi}{2} \cdot \frac{f}{f_0}\right)$$

  - **Note**: At $f = f + 0$ we have $\theta = \pi/2$ rad or 90°, which both correspond to $\lambda_0/4$

Example 2.12: $Z_{\text{in}}^{\text{sc}}$ versus f for $Z_0 = 100\ \Omega$

- Here we put together a simple model in LTspice to see how the impedance (pure reactance here) varies with frequency relative to the quarter-wave frequency

- Here we choose $f_0 = 100$ MHz which has period of $T = 10ns$, so the corresponding tline time delay is $T/4 = 2.5$ ns

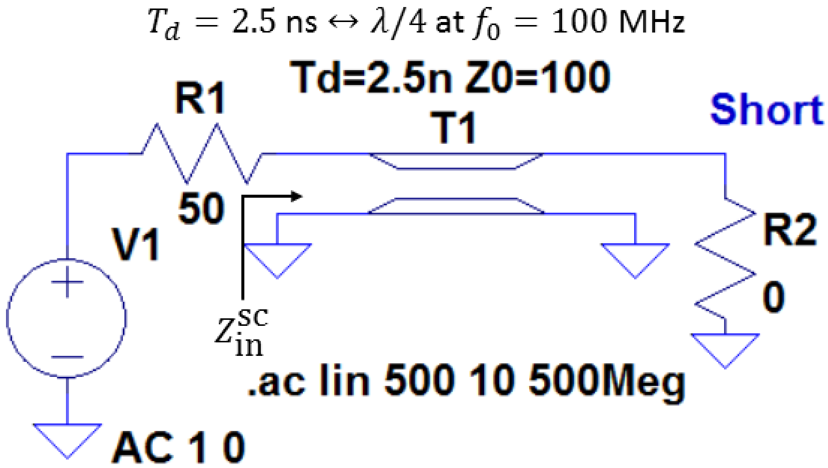


Figure 2.34: LTspice schematic used to study the input impedance versus frequency of a quarter-wave short circuit line at 100 MHz.

- Using a linear plotting scale in LTspice we plot the line input impedance

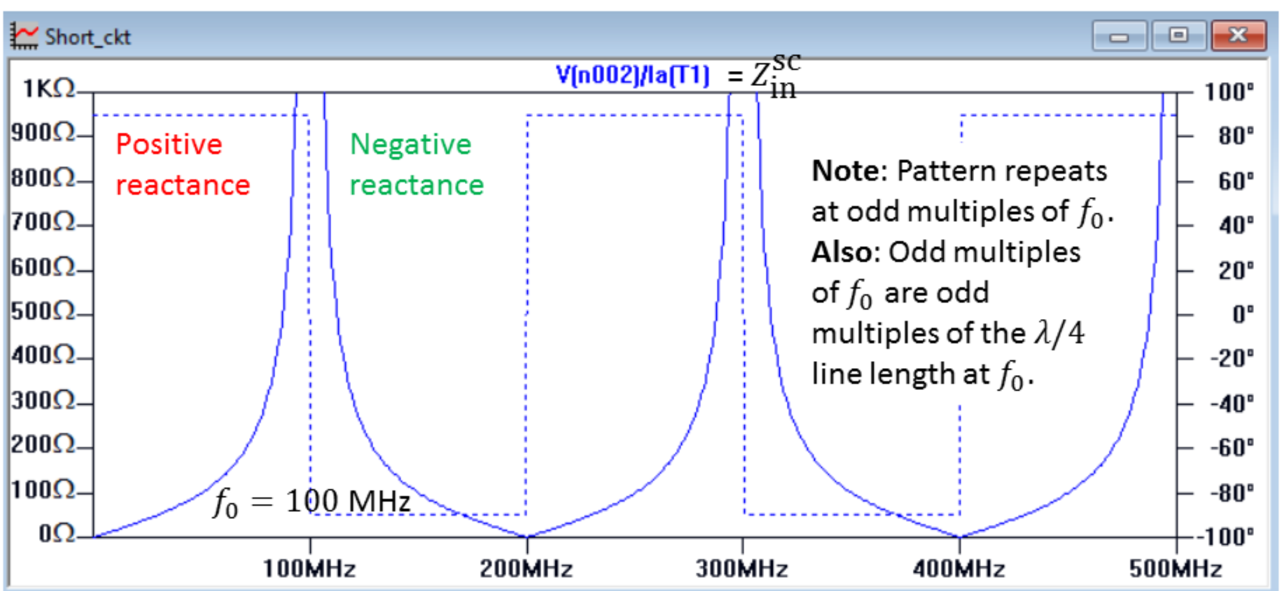


Figure 2.35: Input impedance plot versus frequency, which clearly shows the response repeats at odd multiples of f_0, the quarter-wave frequency.

- **Note**: At the quarter-wave frequency the short circuit appears an open circuit as the impedance magnitude goes to infinity

---

2.8.2 Open-Circuited Line

- For the case of $Z_L = \infty$ the analysis is similar to that of the short circuit

- At distance d away from the load the voltage and current are given by

$$\tilde{V}_{\text{sc}}(d) = V_0^+\left[e^{-j\beta d} + \overbrace{\Gamma}^{+1} e^{-j\beta d}\right] = 2jV_0^+ \cos\beta d$$
$$\tilde{I}_{\text{sc}}(d) = \frac{V_0^+}{Z_0}\left[e^{-j\beta d} - \underbrace{\Gamma}_{+1} e^{-j\beta d}\right] = \frac{2V_0^+}{Z_0}\sin\beta d$$

- The input impedance of a length l line is

$$\boxed{Z_{\text{in}}^{\text{oc}}(l) = Z_{\text{in}}(l)\Big|_{Z_L=\infty} = -jZ_0\cot(\beta l)}$$

- The impedance is purely reactive, but changing with d as we move back from the short circuit load (d increasing)

- The figure below shows that the reactance is negative, as in a capacitor, part of the time and positive, as in a capacitor, at other times

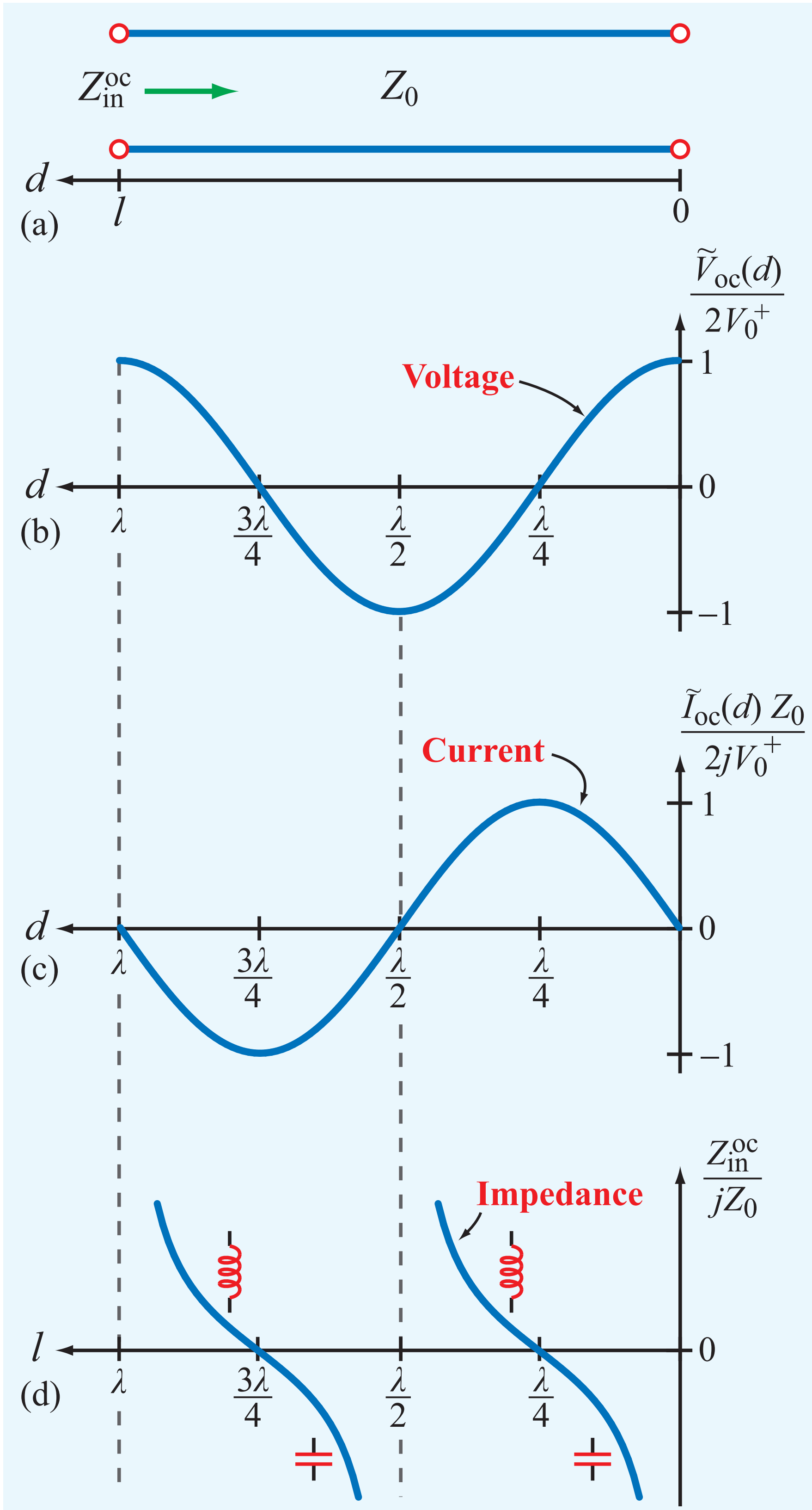


Figure 2.36: Open-circuited line showing voltage, current and $Z_{\mathbf{in}}^{\mathbf{oc}}/jZ_0$ versus l.

Impedance versus Frequency

- We can again view the open circuit impedance as a function of frequency with the line length fixed at some quarter-wave frequency $f_0 \leftrightarrow \lambda_0/4$
- Expressing $Z_{\text{in}}^{\text{oc}}$ as function frequency, we have

$$Z_{\text{in}}^{\text{oc}}(f) = -jZ_0 \cot(\theta) = jZ_0 \cot\left(\frac{\pi}{2} \cdot \frac{f}{f_0}\right)$$

Example 2.13: $Z_{\text{in}}^{\text{oc}}$ versus f for $Z_0 = 100\ \Omega$

- We again put together an LTspice model for the open-circuit case, to see how the impedance (pure reactance here) varies with frequency relative to the quarter-wave frequency
- Here we choose $f_0 = 100$ MHz which has period of $T = 10ns$, so the corresponding tline time delay is $T/4 = 2.5$ ns

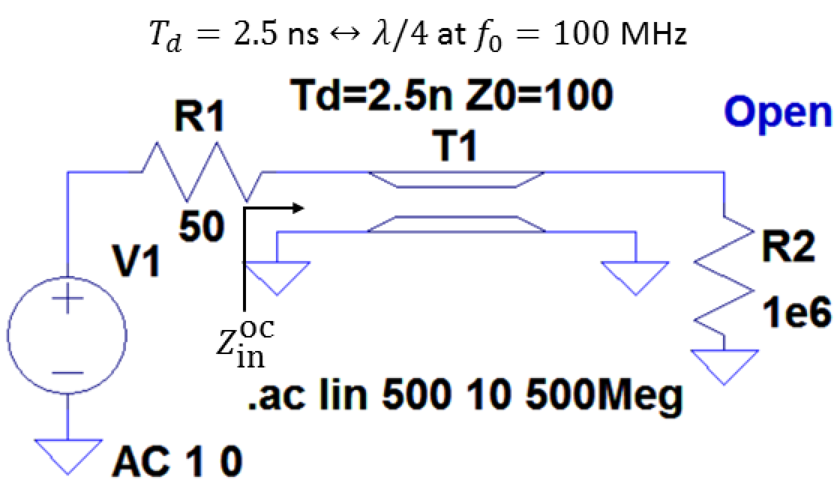


Figure 2.37: LTspice schematic used to study the input impedance versus frequency of a quarter-wave open circuit line at 100 MHz.

- Using a linear plotting scale in LTspice we plot the line input impedance

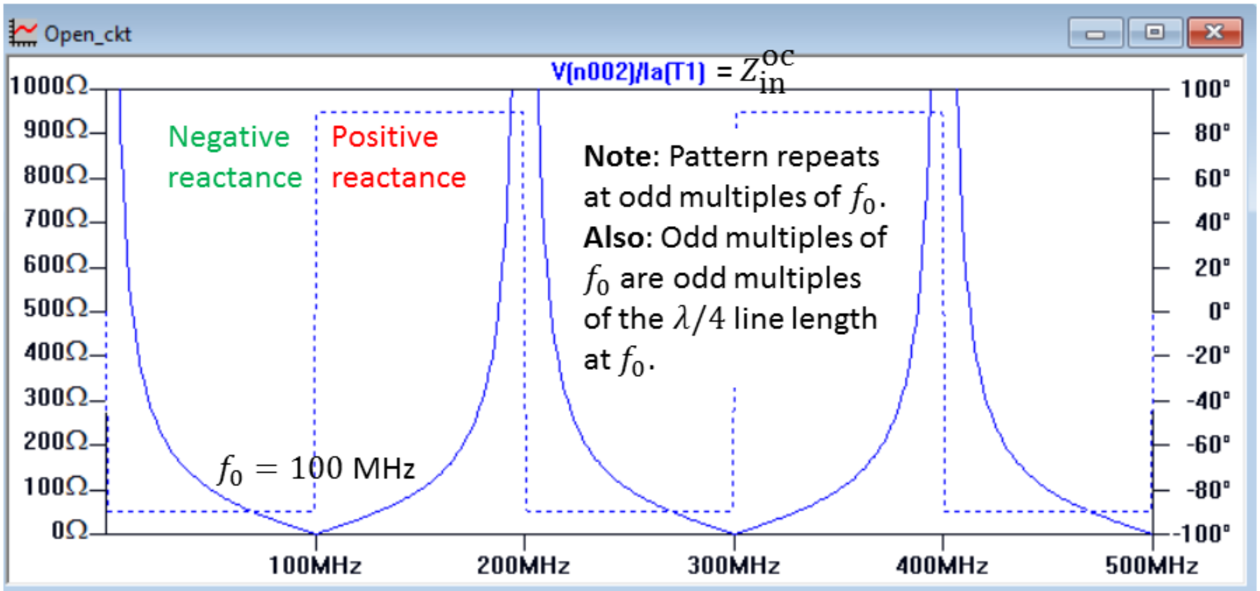


Figure 2.38: Input impedance plot versus frequency, which clearly shows the response repeats at odd multiples of f_0, the quarter-wave frequency.

- **Note**: At the quarter-wave frequency the open circuit appears a short circuit as the impedance magnitude goes to zero

2.8.3 Lines of Length a Multiple of $\lambda/2$

- $\lambda/2$ periodicity has been seen since the first plot of standing waves back in the homework problem of Chapter 1

- With regard to line input impedance this is formalized by considering $l = n\lambda/2$ in $\tan \beta l$, $n = 1, 2, \ldots$

$$\tan(\beta \cdot n\lambda/2) = \tan\big((2\pi/\lambda) \cdot n\lambda/2\big) = \tan(n\pi) = 0$$

- Thus

$$\boxed{Z_{\text{in}} = Z_0 \left[\frac{Z_L + jZ_0 \cdot 0}{Z_0 + jZ_L \cdot 0} \right] = Z_L}$$

- A $\lambda/2$ (and multiples) length line transfers the load impedance to the source end

2.8.4 Quarter-Wavelength Transformer

- Suppose $l = \lambda/4 + n\lambda/2,\ n = 0, 1, \ldots$

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = \frac{\pi}{2},$$

so as $\tan(\pi/2) \to \infty$

$$\boxed{Z_{\textbf{in}} = \frac{Z_0^2}{Z_L},\ l = \frac{\lambda}{4} + n\frac{\lambda}{2},\ n = 0, 1, 2, \ldots}$$

Example 2.14: **Matching a Real Impedance Load (or tline)**

- Suppose a 50 Ω feedline needs to be *matched* to a 100 Ω load

- To eliminate reflections (at a single frequency) we insert a $\lambda/4$ section of transmission line to act as an *impedance transformer*

$$Z_{02} = \sqrt{Z_{\text{in}} \cdot Z_L} = \sqrt{50 \times 100} = 70.7\Omega$$

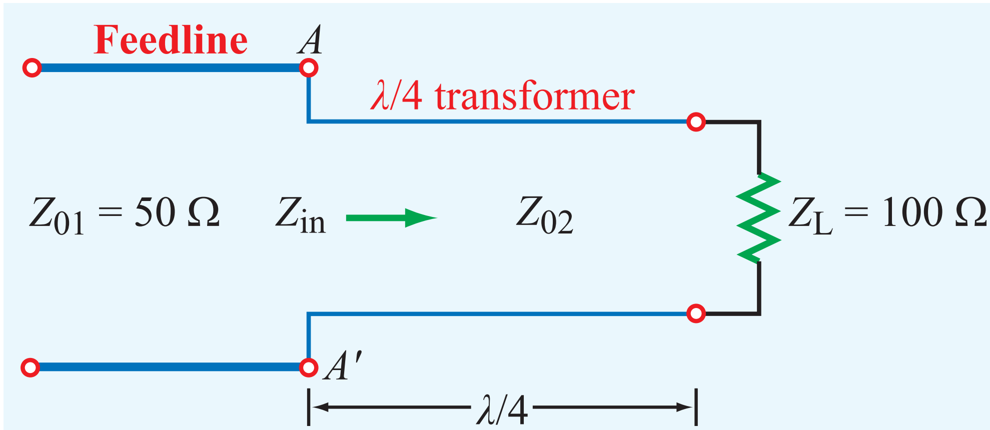


Figure 2.39: Quarter-wave matching of 100 Ω to 50 Ω via $Z_{02} = 70.7\ \Omega$.

- Perfect matching here means that $\Gamma = 0$ looking into the interface at (A, A') is zero
- As the quarter-wave operating frequency changes the matching property deviates from the perfect match ($\Gamma \neq 0$)
- To see this consider an LTspice simulation:

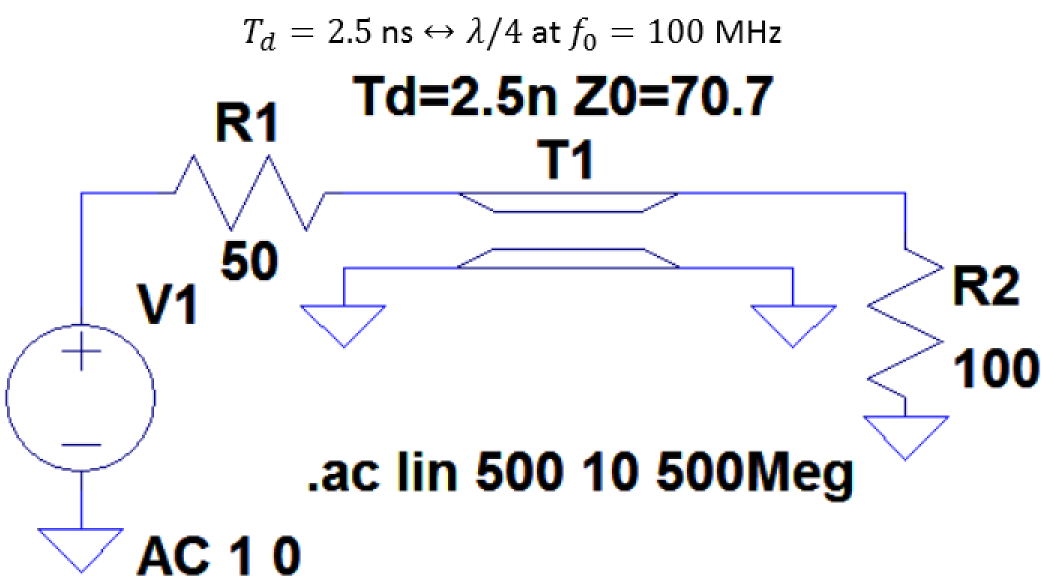


Figure 2.40: LTspice schematic used to study Γ versus frequency of a quarter-wave impedance transformer circuit.

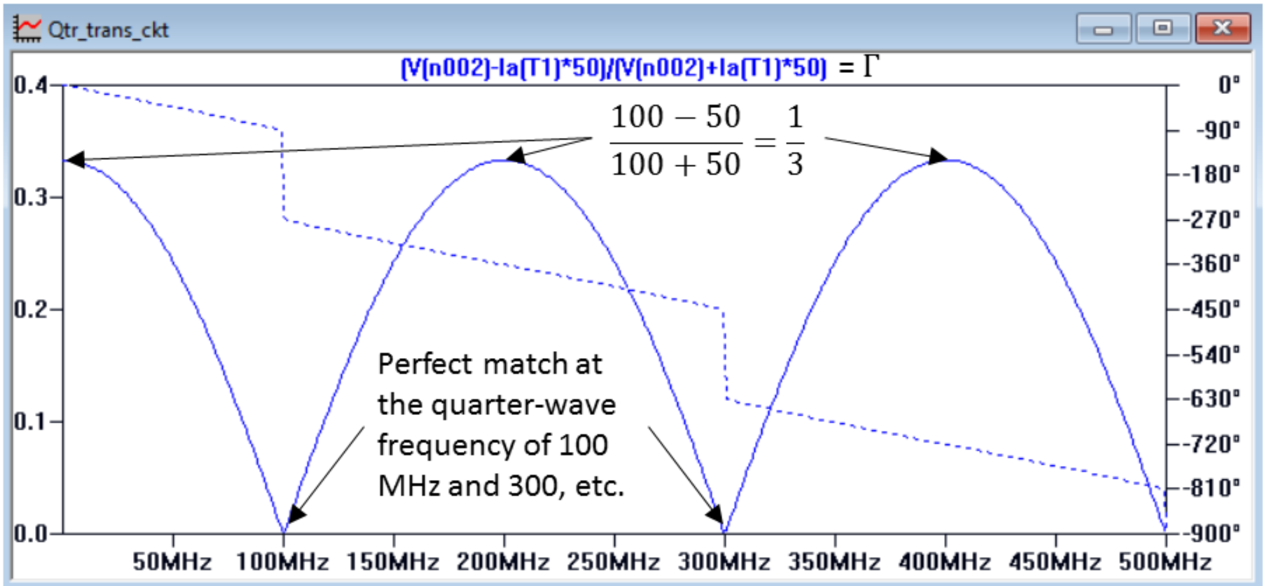


Figure 2.41: Plot of Γ versus frequency of a quarter-wave impedance transformer circuit having $f_0 = 100$ MHz.

- **Note**: At multiples of the $\lambda/2$ frequency the line transfers Z_L to the source end and we simply see the full mismatch of $\Gamma = (100 - 50)/(100 + 50) = 1/3$; surprised?

2.8.5 Matched Line: $Z_L = Z_0$

- When a line is perfectly matched to the load, $Z_L = Z_0$, there is no standing wave pattern and all of the power flowing from the source is delivered to the load; why? ($\Gamma = 0$ for all values of d)

Table 2.5: Summary of standing wave and input impedance phenomina.

Voltage maximum **Voltage minimum**	$\lvert\widetilde{V}\rvert_{\max} = \lvert V_0^+\rvert[1+\lvert\Gamma\rvert]$ $\lvert\widetilde{V}\rvert_{\min} = \lvert V_0^+\rvert[1-\lvert\Gamma\rvert]$
Positions of voltage maxima (also positions of current minima) Position of first maximum (also position of first current minimum)	$d_{\max} = \dfrac{\theta_r\lambda}{4\pi} + \dfrac{n\lambda}{2}, \quad n = 0,1,2,\ldots$ $d_{\max} = \begin{cases} \dfrac{\theta_r\lambda}{4\pi}, & \text{if } 0 \le \theta_r \le \pi \\ \dfrac{\theta_r\lambda}{4\pi} + \dfrac{\lambda}{2}, & \text{if } -\pi \le \theta_r \le 0 \end{cases}$
Positions of voltage minima (also positions of current maxima) Position of first minimum (also position of first current maximum)	$d_{\min} = \dfrac{\theta_r\lambda}{4\pi} + \dfrac{(2n+1)\lambda}{4}, \quad n = 0,1,2,\ldots$ $d_{\min} = \dfrac{\lambda}{4}\left(1 + \dfrac{\theta_r}{\pi}\right)$
Input impedance	$Z_{\text{in}} = Z_0\left(\dfrac{z_L + j\tan\beta l}{1 + jz_L\tan\beta l}\right) = Z_0\left(\dfrac{1+\Gamma_l}{1-\Gamma_l}\right)$
Positions at which Z_{in} is real	at voltage maxima and minima
Z_{in} at voltage maxima	$Z_{\text{in}} = Z_0\left(\dfrac{1+\lvert\Gamma\rvert}{1-\lvert\Gamma\rvert}\right)$
Z_{in} at voltage minima	$Z_{\text{in}} = Z_0\left(\dfrac{1-\lvert\Gamma\rvert}{1+\lvert\Gamma\rvert}\right)$
Z_{in} of short-circuited line	$Z_{\text{in}}^{\text{sc}} = jZ_0\tan\beta l$
Z_{in} of open-circuited line	$Z_{\text{in}}^{\text{oc}} = -jZ_0\cot\beta l$
Z_{in} of line of length $l = n\lambda/2$	$Z_{\text{in}} = Z_L, \quad n = 0,1,2,\ldots$
Z_{in} of line of length $l = \lambda/4 + n\lambda/2$ Z_{in} of matched line	$Z_{\text{in}} = Z_0^2/Z_L, \quad n = 0,1,2,\ldots$ $Z_{\text{in}} = Z_0$

$\lvert V_0^+\rvert$ = amplitude of incident wave; $\Gamma = \lvert\Gamma\rvert e^{j\theta_r}$ with $-\pi < \theta_r < \pi$; θ_r in radians; $\Gamma_l = \Gamma e^{-j2\beta l}$.

2.9 Power Flow in a Lossless Line

- Fundamentally the power flow in a tline is related to the time domain voltage and current $v(d,t)$ and $i(d,t)$
- The phasor quantity $\tilde{V}(d)$ is related to $v(d,t)$ via

$$\begin{aligned} v(d,t) &= \mathrm{Re}\Big[\tilde{V}(d)e^{j\omega t}\Big] \\ &= \mathrm{Re}\Big[|V_0^+|e^{j\phi^+}\big(e^{j\beta d} + |\Gamma|e^{j\phi_r}e^{-j\beta d}\big)\Big] \\ &= |V_0^+|\Big[\cos\big(\omega t + \beta d + \phi^+\big) \\ &\qquad + |\Gamma|\cos(\omega t - \beta d + \phi^+ + \phi_r)\Big] \end{aligned}$$

and similarly for $i(d,t)$

$$\begin{aligned} i(d,t) = \frac{|V_0^+|}{Z_0}\Big[&\cos\big(\omega t + \beta d + \phi^+\big) \\ &- |\Gamma|\cos(\omega t - \beta d + \phi^+ + \phi_r)\Big] \end{aligned}$$

where $V_0^+ = |V_0^+|e^{j\phi^+}$ and $\Gamma = |\Gamma|e^{j\phi_r}$

2.9.1 Instantaneous Power

- Instantaneous power is the product

$$P(d,t) = v(d,t)\cdot i(d,t)$$

which can be written as

$$P(d,t) = P^i(d,t) - P^r(d,t) \quad \text{(book has + sign)}$$

where

$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{2Z_0}\big[1 + \cos(2\omega t + 2\beta d + 2\phi^+)\big]$$
$$P^{\mathrm{r}}(d,t) = |\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}\big[1 + \cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_r)\big]$$

(book has minus on reflected power term)

- In the average power calculation, up next, the 2ω terms average to zero

2.9.2 Time-Averaged Power

- From circuits and systems, the time averaged power is the primary interest
- For a periodic signal of period $T = 2\pi/\omega$, the average power in W is

$$P_{\mathrm{av}}(d) = \frac{1}{T}\int_0^T P(d,t)\,dt$$

- The double frequency term will average to zero (why?) yielding

$$\begin{aligned} P_{\mathrm{av}} &= \overbrace{\frac{|V_0^+|^2}{2Z_0}}^{P_{\mathrm{av}}^{\mathrm{i}}} - \overbrace{|\Gamma|^2\frac{|V_0^+|^2}{2Z_0}}^{P_{\mathrm{av}}^{\mathrm{r}}} \\ &= \frac{|V_0^+|^2}{2Z_0}\big[1 - |\Gamma|^2\big] \quad \text{(W)} \end{aligned}$$

- **Note**: The power flow is independent of d for both the incident and reflected terms

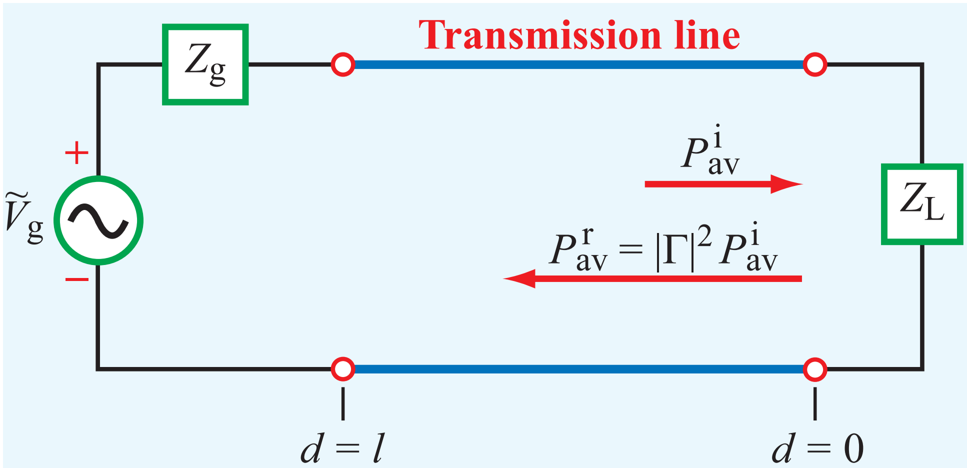


Figure 2.42: Summary of power flow in a loss transmission line.

Phasor Domain Power Calculation

- Direct calculation of the power in the phasor domain is possible using

$$\boxed{P_{av} = \frac{1}{2}\text{Re}\left[\tilde{V} \cdot \tilde{I}^*\right]}$$

Example 2.15: **Reflected Power**

- Given 100 (W) of power sent down a tline toward the load and $|\Gamma| = 0.2$, how much power is returned?
- According to the average power expressions,

$$P^r_{av} = |\Gamma|^2 P^i_{av} = 0.04 \times 100 = 4 \quad \text{(W)}$$

2.10 The Smith Chart

The *Smith chart*, developed in 1939 by P.H. Smith, is a widely used graphical tool for working and visualizing transmission line theory problems.

- In mathematical terms the Smith chart is a transformation between an impedance Z and the reflection coefficient Γ, by virtue of the relation

$$\Gamma = \frac{Z - Z_0}{Z + Z_0}$$

- In complex variable theory this is known as a *bilinear transformation*

- Assuming Z_0 is real the transformation maps the Z-plane to the Γ-plane as shown below:

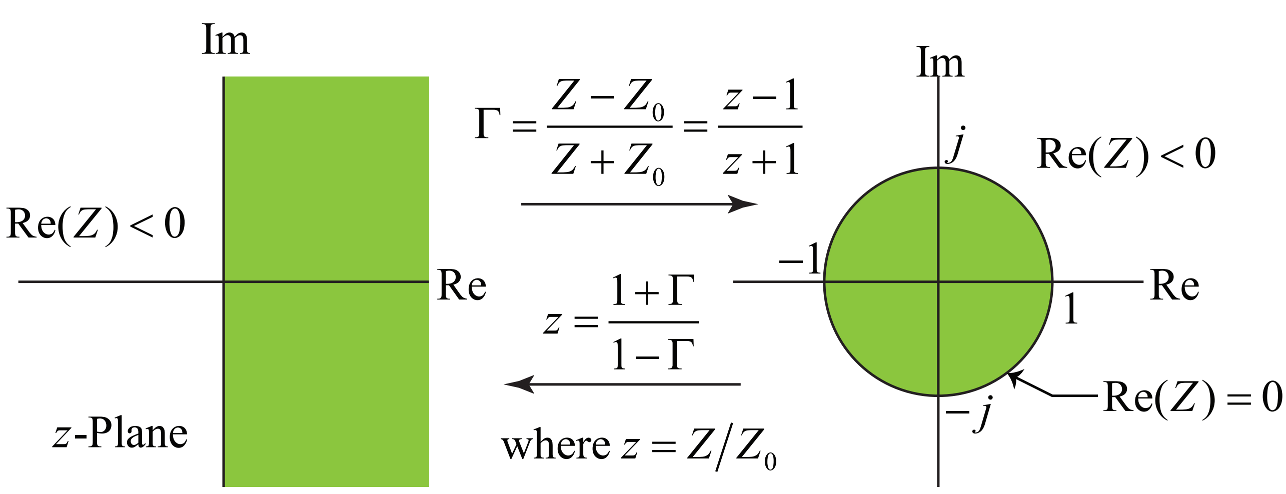


Figure 2.43: Smith chart as a mapping from Z to Γ.

- When working with the Smith chart impedances are normalized by Z_0, i.e. $z = Z/Z_0$, so

$$\Gamma = \frac{z-1}{z+1} \quad \text{and} \quad z = \frac{1+\Gamma}{1-\Gamma}$$

2.10.1 Parametric Equations and the Γ-Plane

- To better understand the transformation let $z = r + jx$

$$\Gamma = \text{Re}(\Gamma) + j\text{Im}(\Gamma) = U + jV = \frac{(r-1) + jx}{(r+1)^2 + jx}$$

- The real and imaginary parts are

$$U = \frac{r^2 - 1 + x^2}{(r+1)^2 + x^2}$$
$$V = \frac{2x}{(r+1)^2 + x^2}$$

- Now, eliminate x from the above expressions for U and V

$$\left(U - \frac{r}{r+1}\right)^2 + V^2 = \left(\frac{1}{r+1}\right)^2$$

- Thus for a *constant* r we have a circle of radius $1/(r+1)$ in the $U - V$ plane (Γ-plane) with center at

$$(U_0, V_0) = (r/(r+1), 0)$$

- Putting the two circle types together results the high-level Smith chart shown below

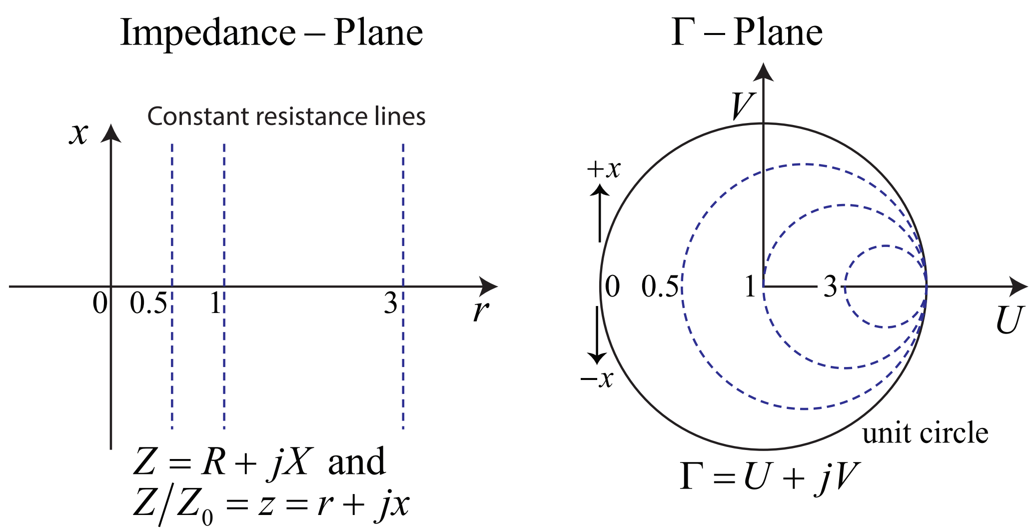


Figure 2.44: Constant r circles in the Γ-plane.

- By eliminating r from the U and V equations we find

$$(U-1)^2 + (V - \frac{1}{x}) = \left(\frac{1}{x}\right)^2$$

- Thus for *constant* x we have a circle of radius $1/x$ in the $U - V$ plane with center at

$$(U_0, V_0) = (1, 1/x)$$

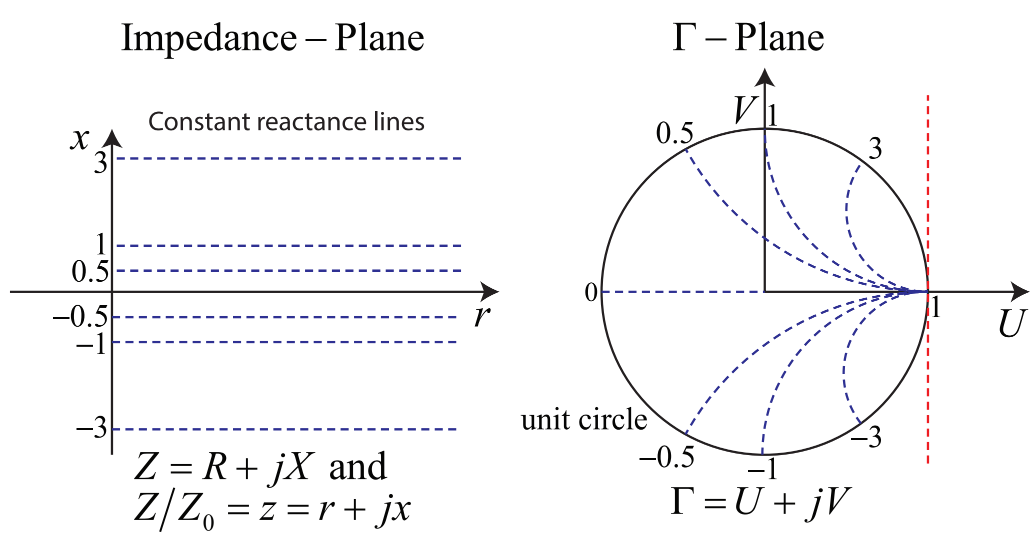


Figure 2.45: Constant x circles in the Γ-plane.

- Next place the constant resistance and constant reactance circles on the same unit circle Γ-plane

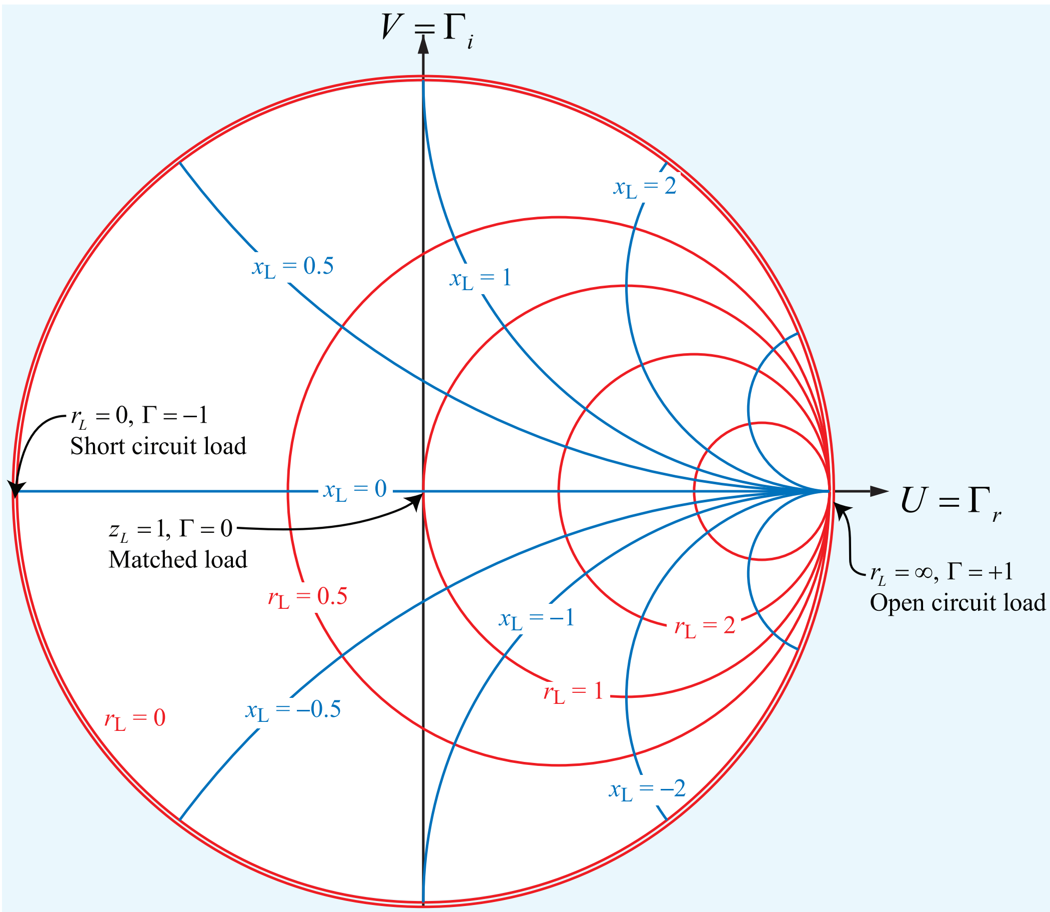


Figure 2.46: A few constant r and constant x circles on he unit circle Γ-plane.

Example 2.16: **Getting Acquainted**

- Suppose that $Z_L = 50 + j50$ and find Z_{in} and S for a line having electrical length of 45° (recall $\beta d = \theta$) and $Z_0 = 50\ \Omega$

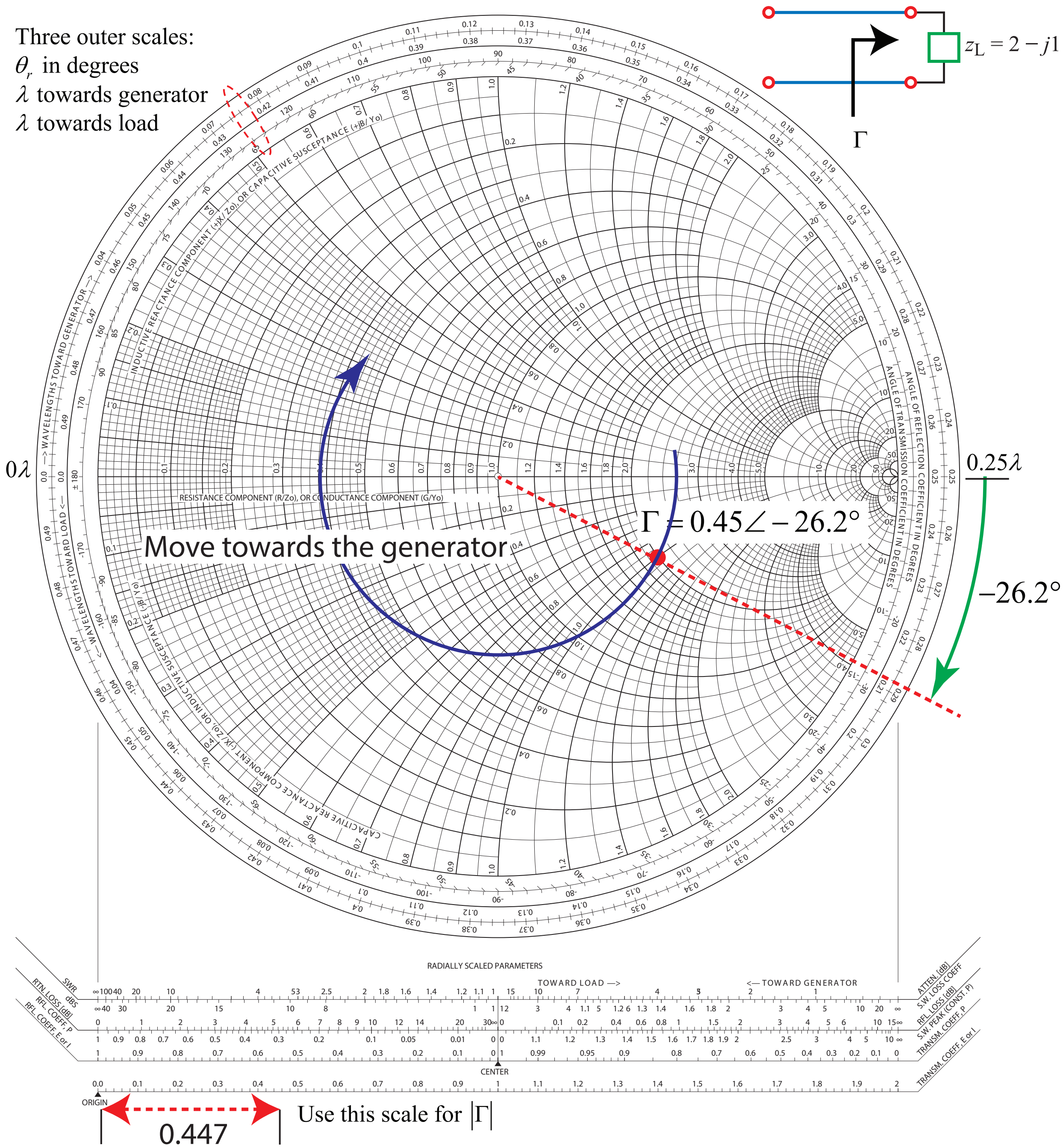


Figure 2.47: Pointing out some of the scales found on the Smith chart.

2.10.2 Wave Impedance

- Transmission calculations using the Smith chart follow from the basic lossless line result that the reflection coefficient looking into a length d line is

$$\Gamma_d = \Gamma e^{-j2\beta d} = \Gamma e^{j(\theta_r - 2\beta_d)},$$

where as before $\Gamma = |\Gamma| e^{j\theta_r}$ is the reflection coefficient of the load

  - Think of Γ as a vector in the Γ-plane, with origin at the chart center $(U, V) = (0, 0)$
  - To find Γ_d you rotate the tip of the Γ *vector* clockwise by $\beta d = \theta$ degrees, as a negative angle in the complex plane is indeed clockwise
  - In Smith chart terminology this is called *moving towards the generator*
  - Moving in the opposite direction, positive angle shift of the Γ vector, is called *moving towards the load*

- Another impact of the $2\beta d$ factor is that $d = \lambda/2$ is one trip around the chart (follows from $\lambda/2$ periodicity of tline input impedance)

- Once Γ_d is found, it then follows that

$$z(d) = \frac{Z(d)}{Z_0} = \frac{1 + \Gamma_d}{1 - \Gamma_d}$$

- Additionally, since
$$S = \text{VSWR} = \frac{1 + |\Gamma|}{1 - \Gamma|}$$
traces out a *constant VSWR* circle in the Γ-plane

- There are scales on the chart for directly reading S or the $r > 1$ axis scale can also be used (see next example)

Example 2.17: **Find Z_{in}**

- Suppose that $Z_L = 50 + j50$ and find Z_{in} and S for a line having electrical length of 45° (recall $\beta d = \theta$) and $Z_0 = 50\ \Omega$
- Normalize by $Z_0 = 50$ we plot $z_L = 1 + j1$ on the chart
- Next we note the wavelengths toward generator value; here it is 0.162λ
- Draw a line through the origin that passes through the wavelengths toward generator value of $0.162\lambda + 0.125\lambda = 0.287\lambda$, this is angle location of Γ_d
- Draw a circle centered on the origin that passes through z_L and rotates clockwise to intersect the radial line passing through 0.287λ
- The intersection point is $z(d = 0.125\lambda) = 2 - j1$
- The value $Z_{\text{in}} = Z_0 \cdot z(0.125\lambda) = 100 - j50$
- The VSWR is read from the $r > 1$ axis to be 2.62; note the voltage minima can be read from the $r < 1$ axis

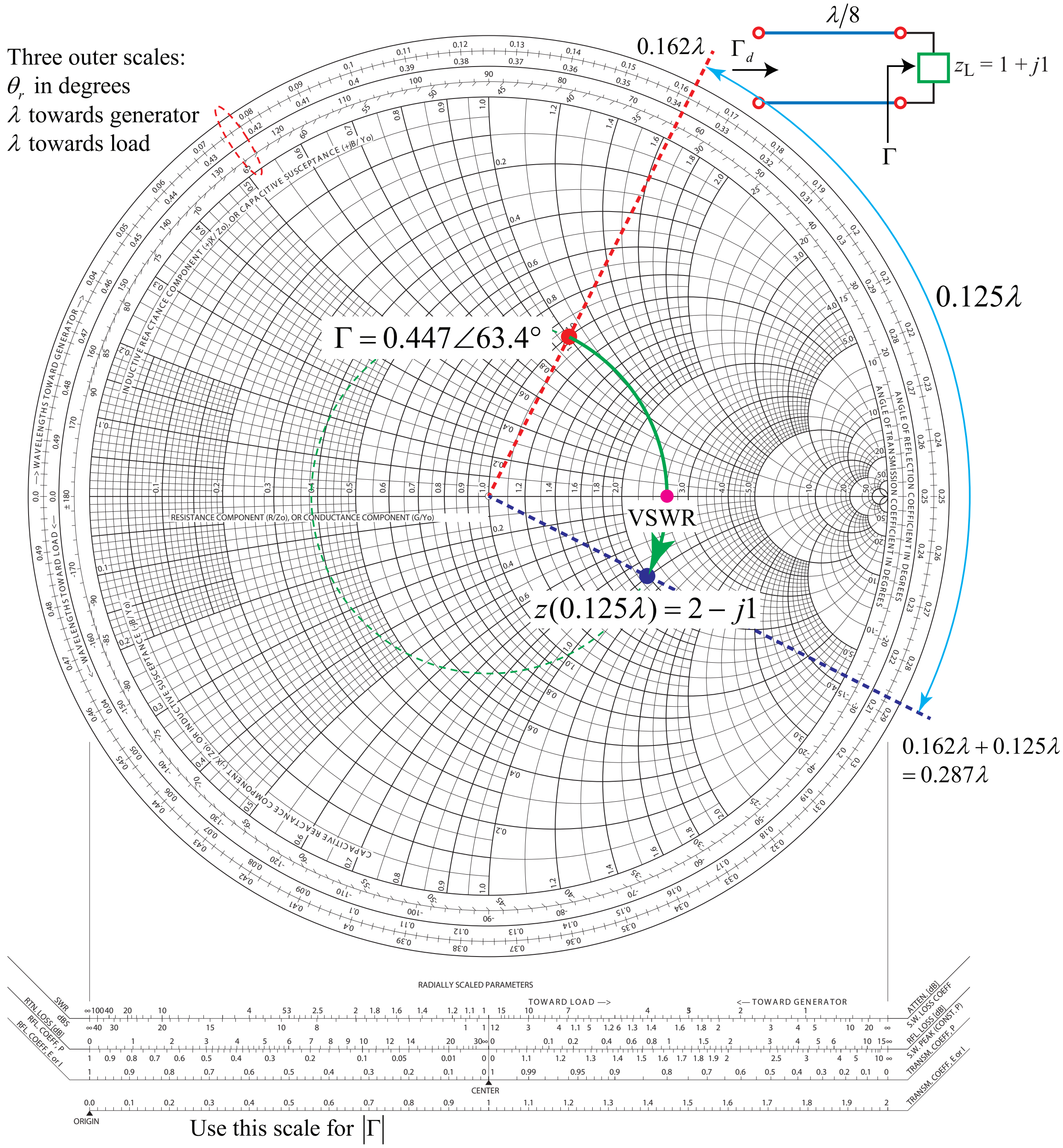


Figure 2.48: Finding $z(d = \lambda/8)$ when $z_L = 1 + j1$.

2.10.3 Impedance/Admittance Transformation

- Converting an impedance to an admittance is simply a matter of rotating the normalized impedance value, z plotted on the chart by 180°, then de-normalizing by multiplying the value found, y, by $Y_0 = 1/Z_0$

- To prove this consider …(see book)

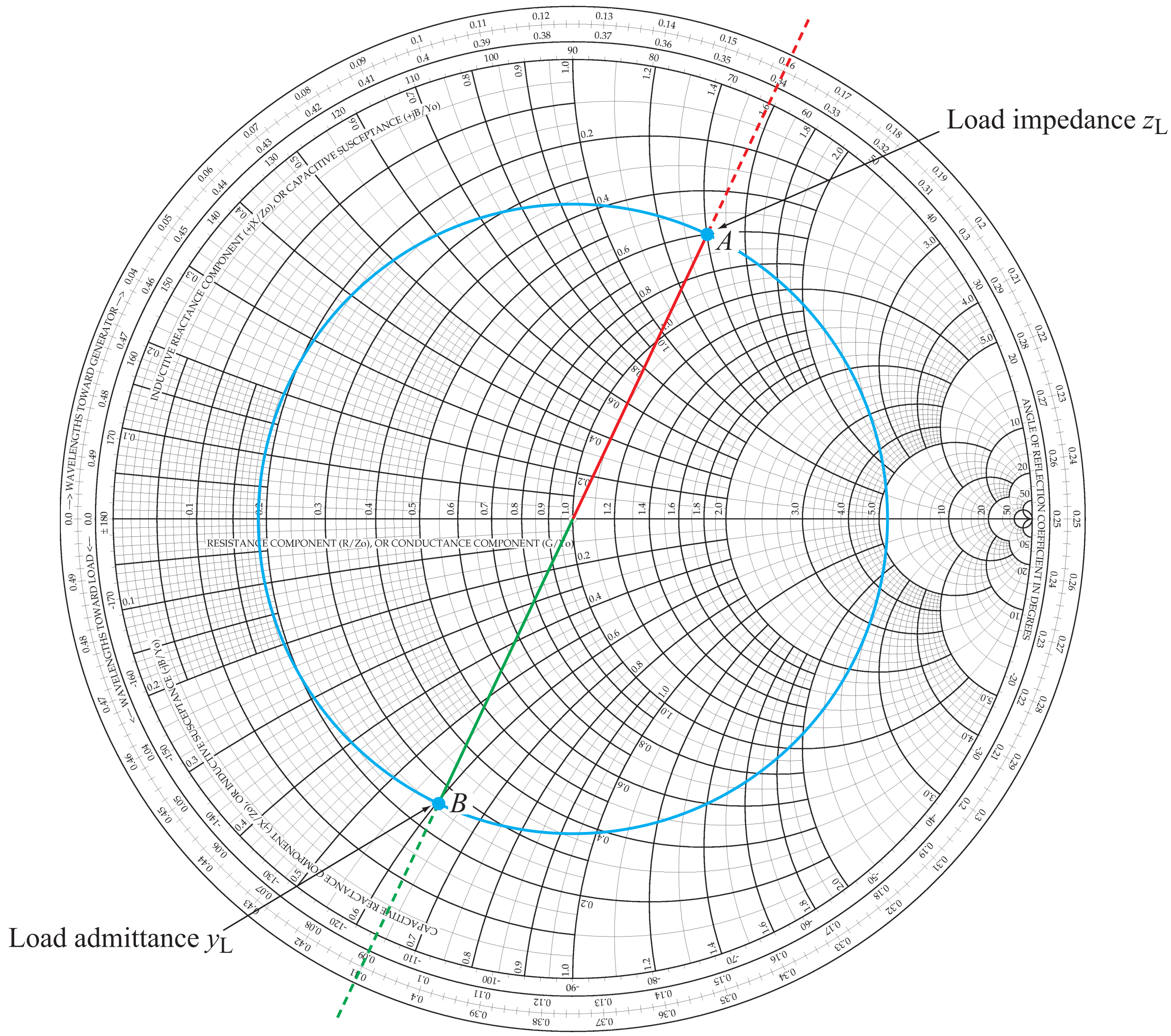


Figure 2.49: Converting $z_L = 0.6 + j1.4$ to $y_L = 0.25 - j0.6$.

2.11 Impedance Matching

- The Smith chart is often used to design impedance matching circuits

- By impedance matching, we mean making the load Z_L appear to be Z_0, like in the design of the quarterwave transformer

- Here the load impedance is in general complex

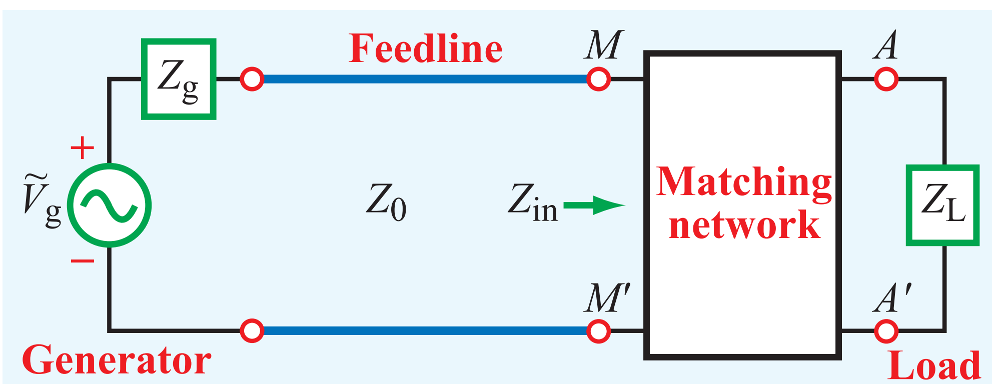


Figure 2.50: The general impedance matching problem.

- The circuit elements used n a matching network may be lumped L and C and/or lossless transmission line sections and/or open and short circuit stubs that attach to the through circuit

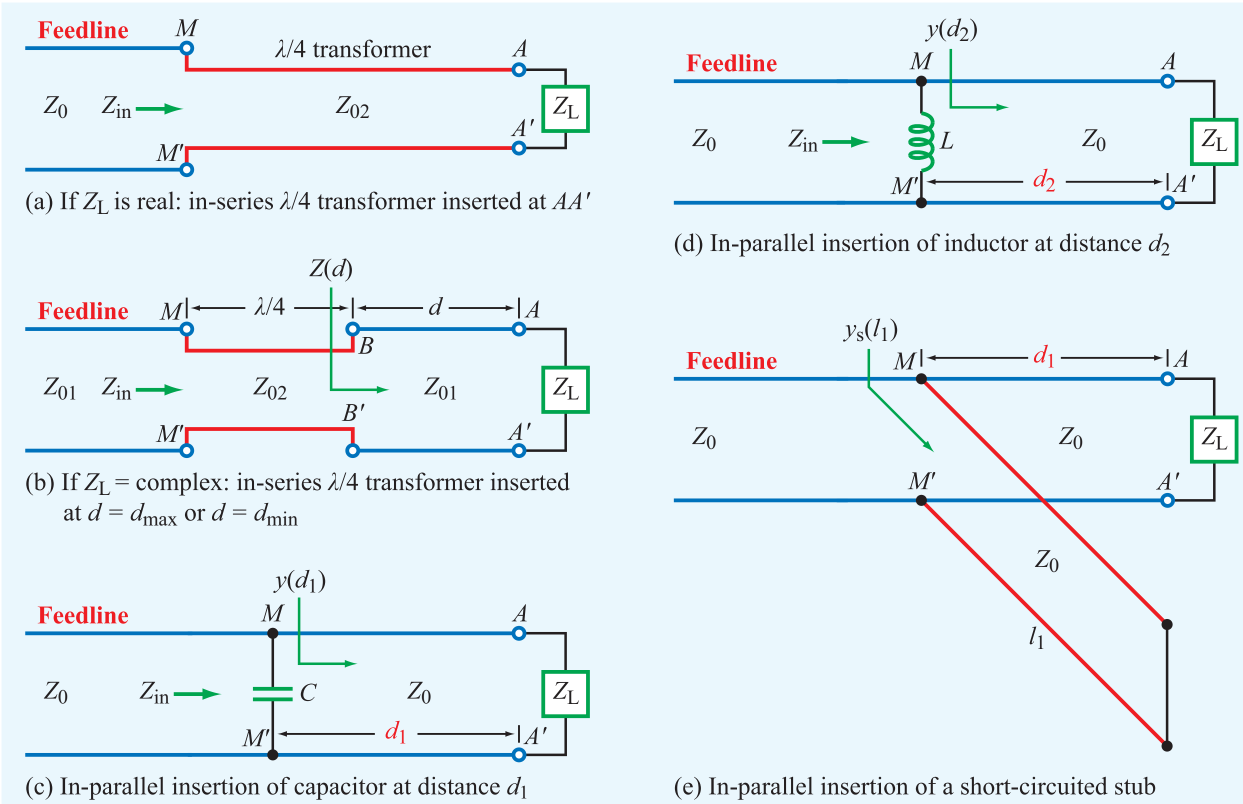


Figure 2.51: A collection of matching circuits.

- Working through all the possible combinations is beyond the scope of this class
- Here we will explore a few examples
- The goal in all cases is to drive the impedance seen looking into the matching circuit to be Z_0 on the Smith chart, that is $\Gamma = 0 + j0$, also known as the *match point*
- The Smith always find a graphical solution, but pure analytical solutions are also possible
- For the network topologies chosen here there are two possible solutions per approach

2.11.1 Quarter-wave Transformer Matching

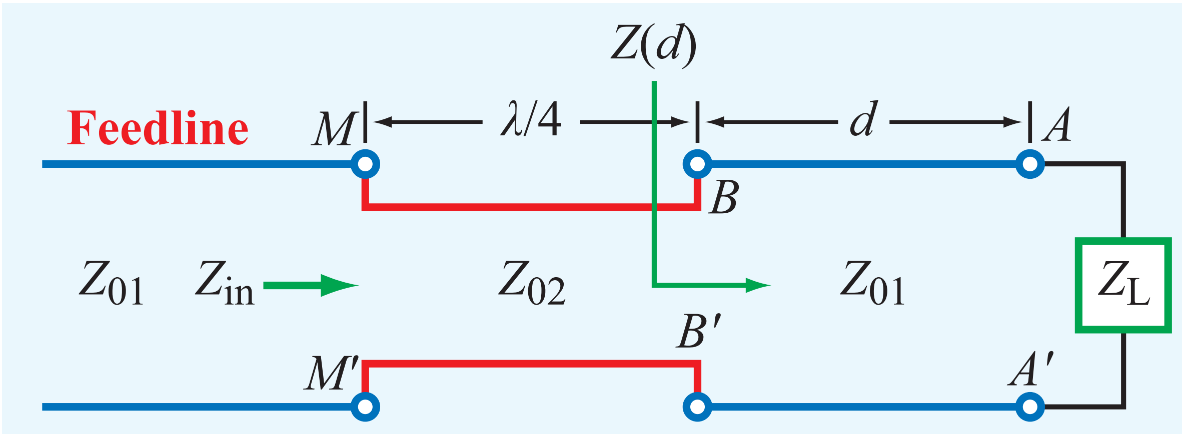


Figure 2.52: A quarter-wave matching approach where $d = d_{\text{max}}$ or d_{min} for complex-to-real conversion.

Analytical Solution

- For the network topology shown above, originally in 2.50 circuits (a), you start from the load, Z_L and *move toward the generator* until either d_{min} or $d_{\mathbf{max}}$ is reached
  - Yes, there are two possible solutions
  - **Recall**: d_{max} corresponds to the first voltage maximum along the line and d_{min} corresponds to the first voltage minimum along the line

- From earlier analysis

$$d_{\text{max}} = \begin{cases} \left(\frac{\theta_r^\circ}{720^\circ}\right)\lambda, & 0 \le \theta_r^\circ < 180^\circ \\ \left(\frac{1}{2} + \frac{\theta_r^\circ}{720^\circ}\right)\lambda, & -180^\circ \le \theta_r^\circ < 0 \end{cases}$$

$$d_{\text{min}} = \left(\frac{1}{4} + \frac{\theta_r^\circ}{720^\circ}\right)\lambda$$

- The next and final step is find the quarter-wave section characteristic impedance
- Choose

$$Z_{02} = \sqrt{Z_{01} \cdot Z(d)} = \sqrt{Z_{01} \cdot z(d) Z_{01}} = Z_{01}\sqrt{z(d)}$$

- The normalized impedance $z(d)$ corresponds to S = VSWR: for $d_{\max}$ it is S, while for $d_{\min}$ it is $1/S$ (this is very obvious on the Smith chart)
- In summary:

$$Z_{02} = \begin{cases} Z_{01}\sqrt{S}, & d = d_{\max} \\ Z_{01}\sqrt{1/S}, & d = d_{\min} \end{cases}$$

  - **Recall**: $S = (1 + |\Gamma|)/(1 - |\Gamma|)$

Example 2.18: A series RC Load

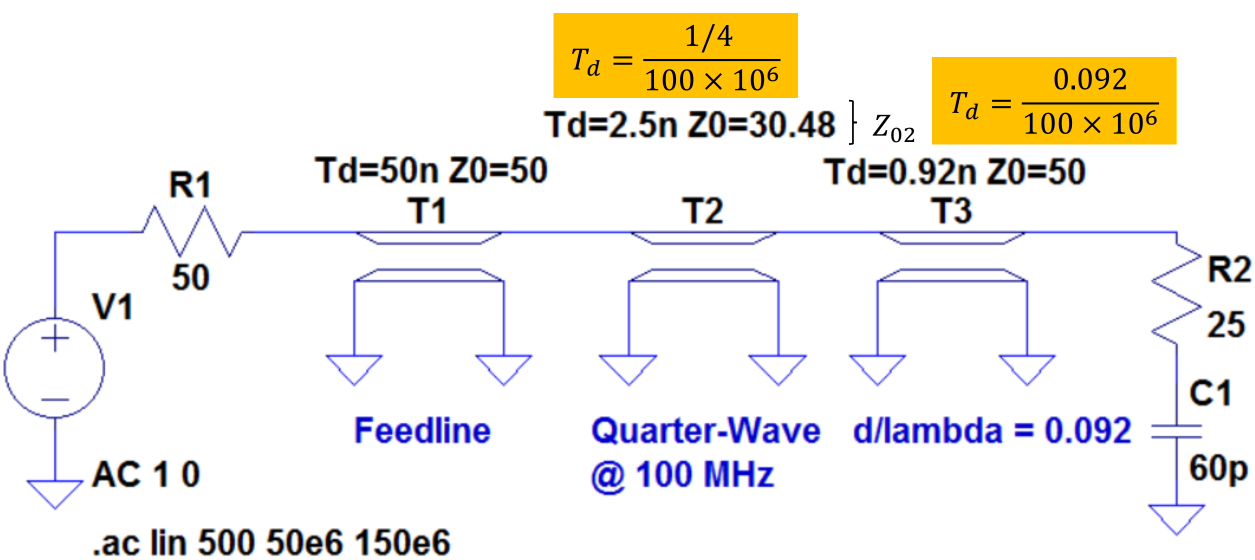


Figure 2.53: LTspice schematic of the $d_{\min}$ solution.

- The load consists of $R = 25\ \Omega$ in series with $C = 60$ pF capacitor
- The match is to be made at $f = 100$ MHz which has a period of 10 ns
- **Solution 1**: Choose the d_{min} case and use Python to obtain the numerical values

```
Z0 = 50
Cs = 60e-12 # C = 60 pF cap
ZL = 25 + 1/(1j*2*pi*100e6*Cs)
print('ZL = ' + cpx_fmt(ZL))
Gamma = ZL2Gamma(ZL,Z0)
print('Gamma = ' + cpx_fmt(Gamma,'polar'))
```

```
ZL = 25.0000 - j26.5258
Gamma = 0.4582 /_-113.8263 (deg)
```

Solution 1: Based on d_{min}

```
d_min = 1/4 + angle(Gamma)*(180/pi)/720
print('d_min/lambda = %1.3f' % d_min)
```

```
d_min/lambda = 0.092
```

```
S = (1 + abs(Gamma))/(1 - abs(Gamma))
Z_02 = Z0*sqrt(1/S)
print('Z_02 = %1.2f (Ohms)' % Z_02)
```

```
Z_02 = 30.48 (Ohms)
```

Figure 2.54: Solution 1 calculation in a Jupyter notebook.

- The design values found above are used in the original LTspice schematic shown earlier
- A frequency response plot of $20\log_{10}|\Gamma_{\text{in}}|$ is obtained and compared with $20\log_{10}|\Gamma|$, which is the no matching case
  - **Note**: This quantity is related to the *Return Loss* of a one-port network

– Formally,

$$\text{Return Loss} = \text{RL} \equiv -20\log_{10}|\Gamma| \quad \text{(dB)}$$

– Loss is always a positive quantity, but in LTspice it is more convenient to plot the negative of RL

- Here the 20 dB RL bandwidth is found to be $\sim$ 10.34 MHz

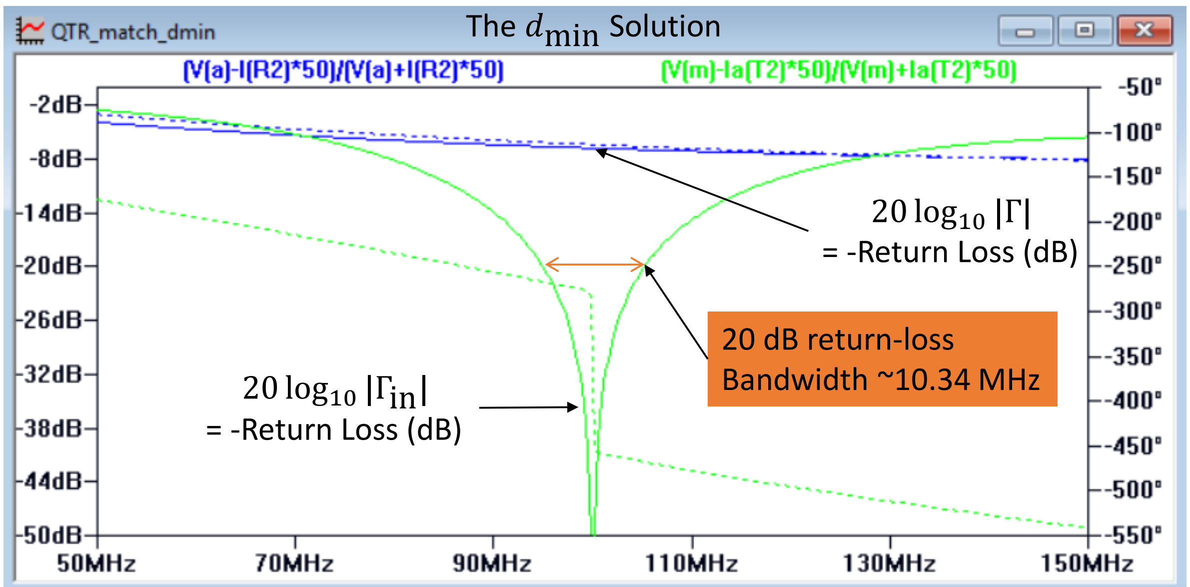


Figure 2.55: Frequency response plot of solution 1 as the negative of the Return Loss

- At 100 MHz the match is perfect as RT approaches ∞ (in theory with no component error)

- Without any matching circuit RT is about 5 dB, very poor

- Moving on to solution 2, the next three figures show the design, the final schematic, and the frequency response

Solution 2: Based on $d_{\max}$

```
d_max = mod(angle(Gamma)*(180/pi)/720,0.5)
print('d_max/lambda = %1.3f' % d_max)
```

```
d_max/lambda = 0.342
```

```
S = (1 + abs(Gamma))/(1 - abs(Gamma))
Z_02 = Z0*sqrt(S)
print('Z_02 = %1.2f (Ohms)' % Z_02)
```

```
Z_02 = 82.03 (Ohms)
```

Figure 2.56: Solution 2 calculation in a Jupyter notebook.

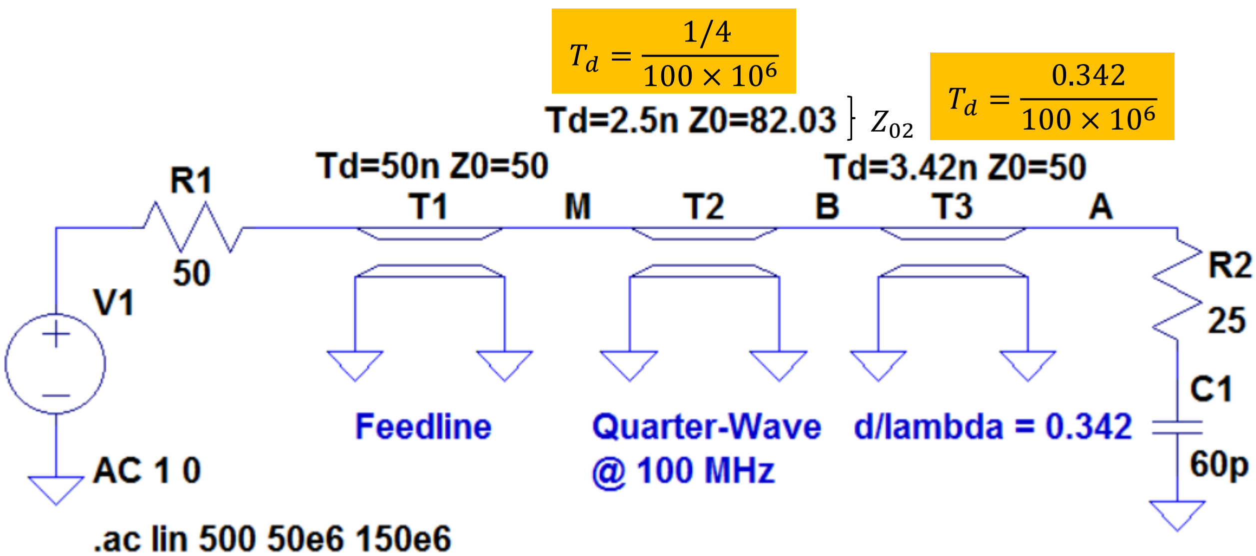


Figure 2.57: Solution 2 LTspice schematic for quarter-wave matching.

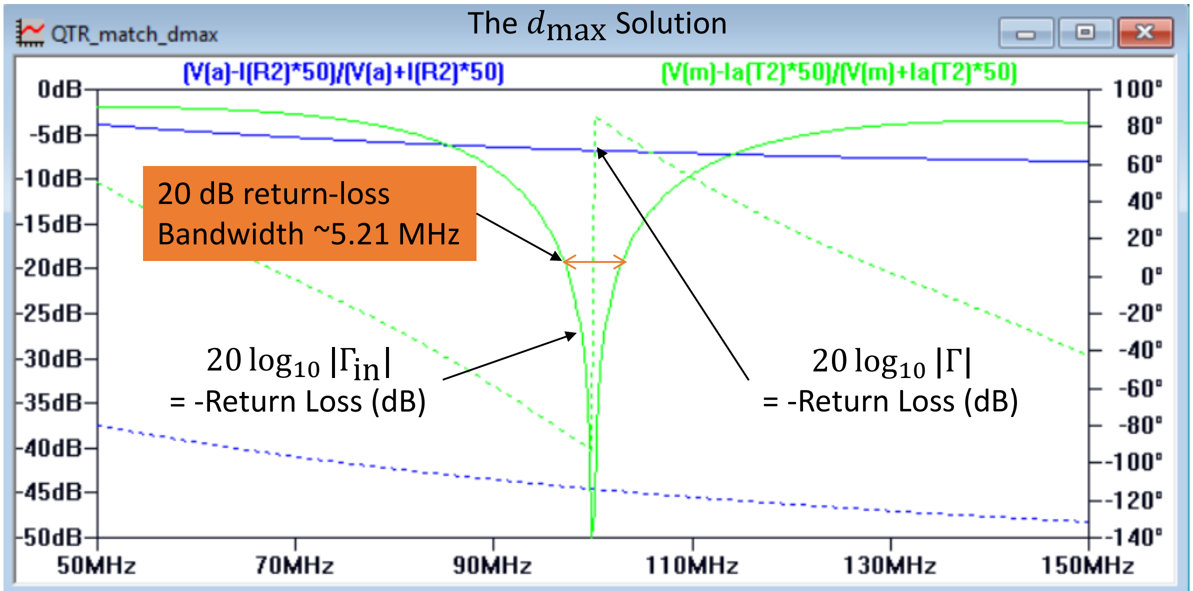


Figure 2.58: Frequency response plot of solution 2 as the negative of the Return Loss

- Here the RL bandwidth is only 5.21 MHz; likely due to $d_{\max}$ being a $\lambda/2$ longer line section

Smith Chart Solution

- The tutorial of text Module 2.7 does a great job of explaining the steps

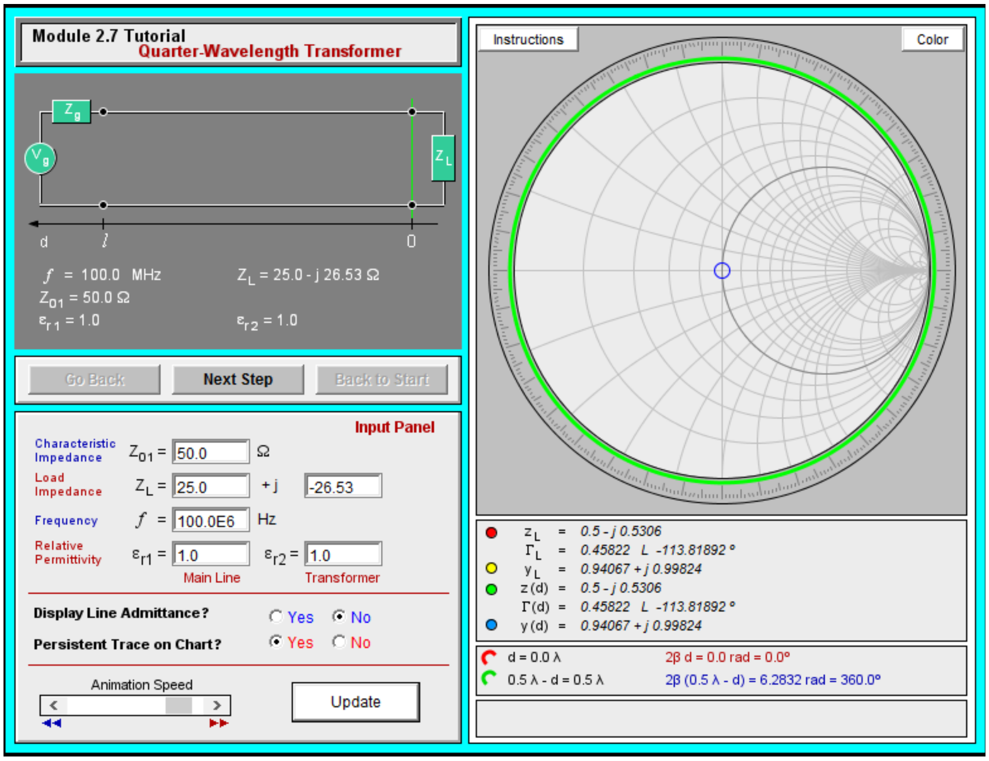


Figure 2.59: Quarter-wave match Smith chart: Applet set-up

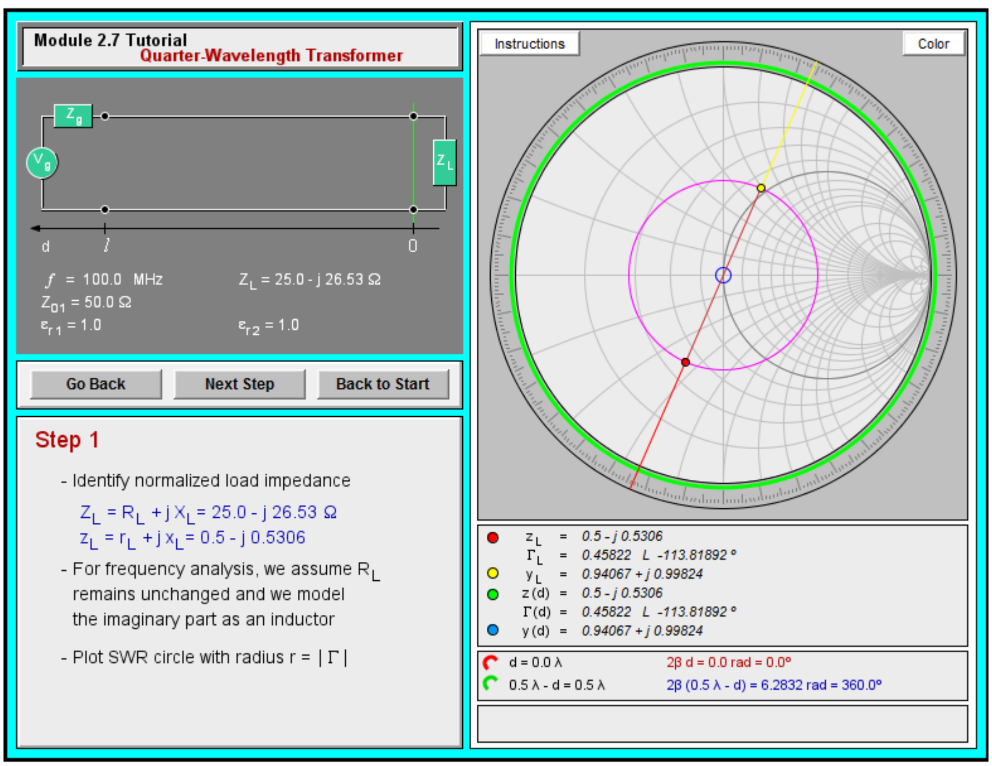


Figure 2.60: Quarter-wave match Smith chart: Step 1

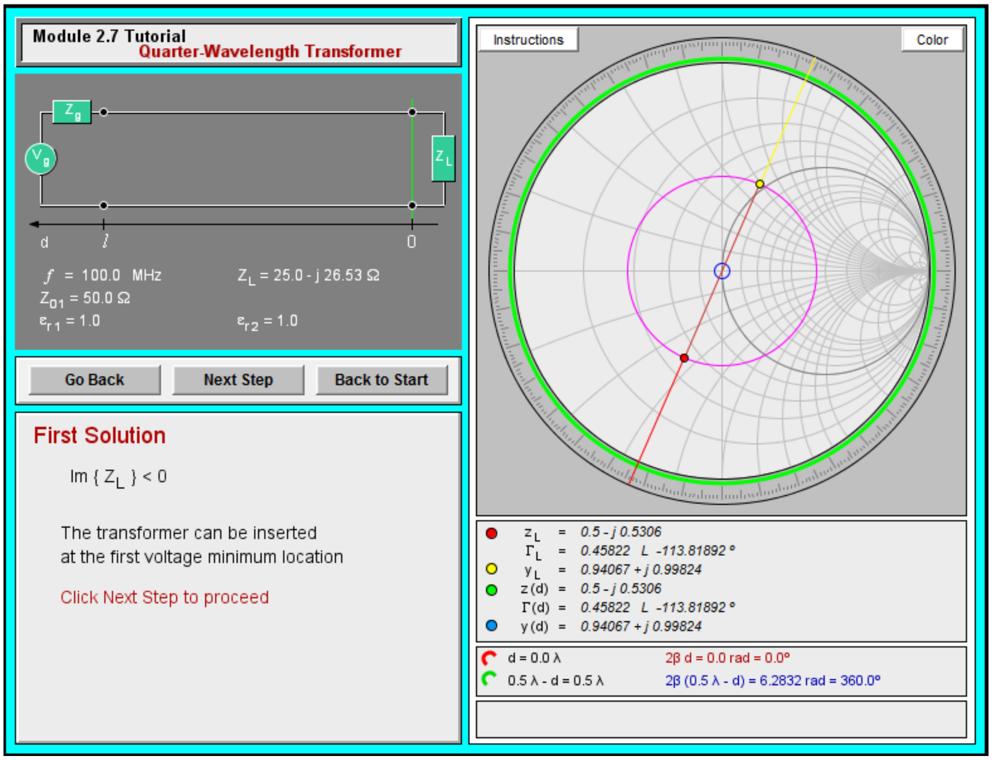


Figure 2.61: Quarter-wave match Smith chart: Solution 1

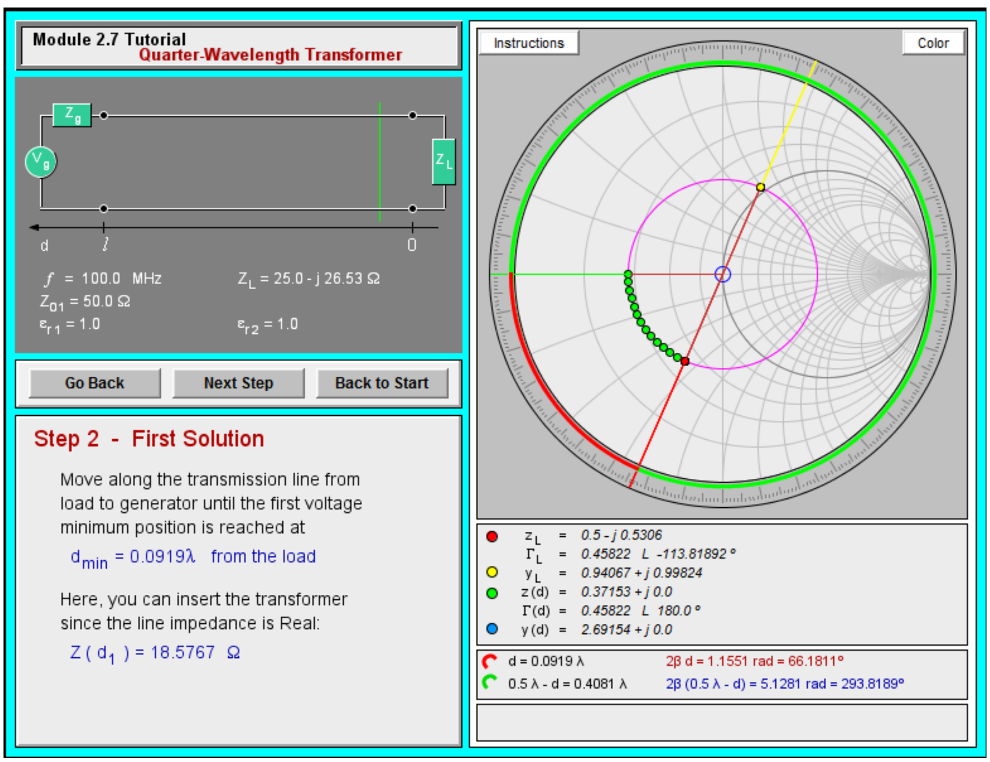


Figure 2.62: Quarter-wave match Smith chart: Step 2

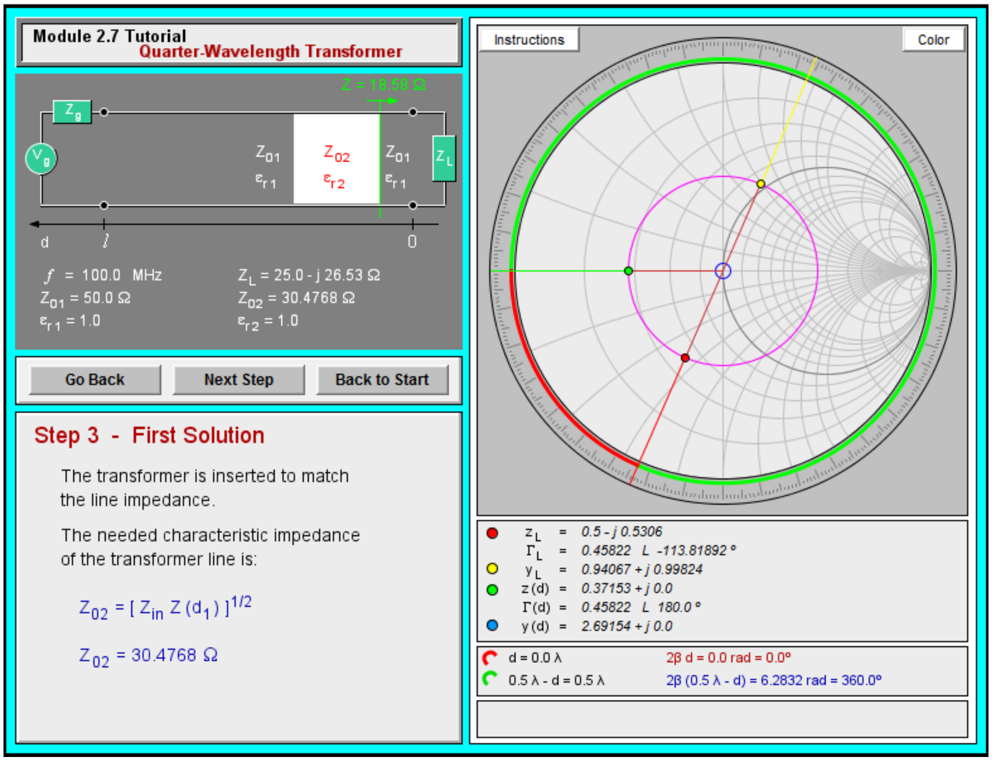


Figure 2.63: Quarter-wave match Smith chart: Step 3

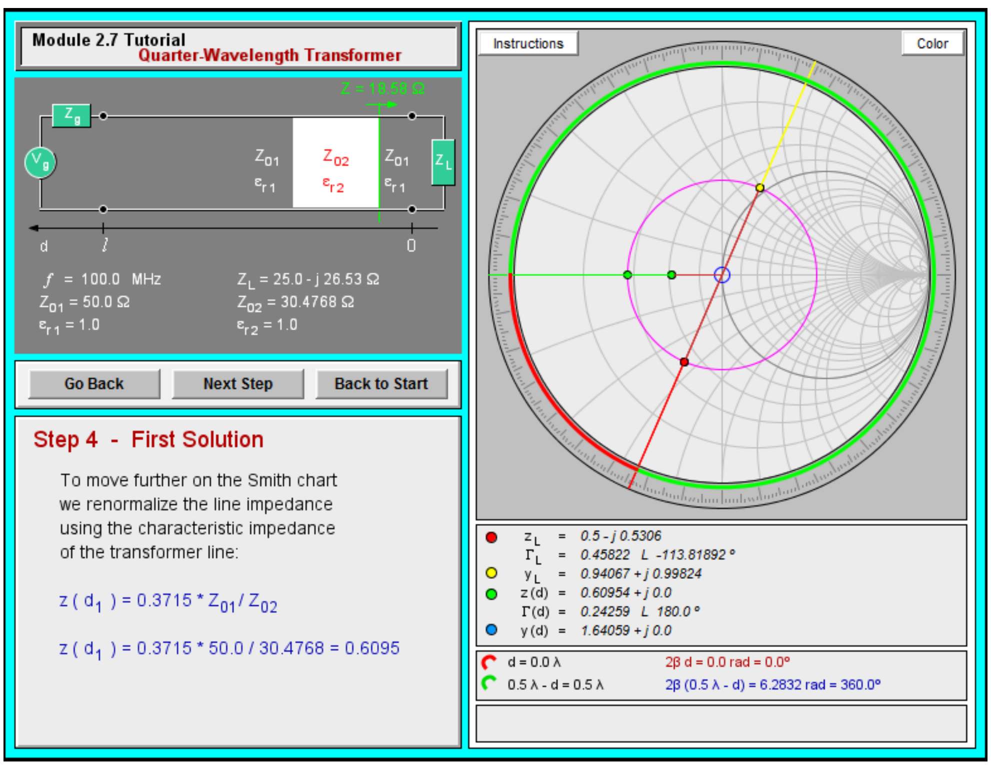


Figure 2.64: Quarter-wave match Smith chart: Step 4

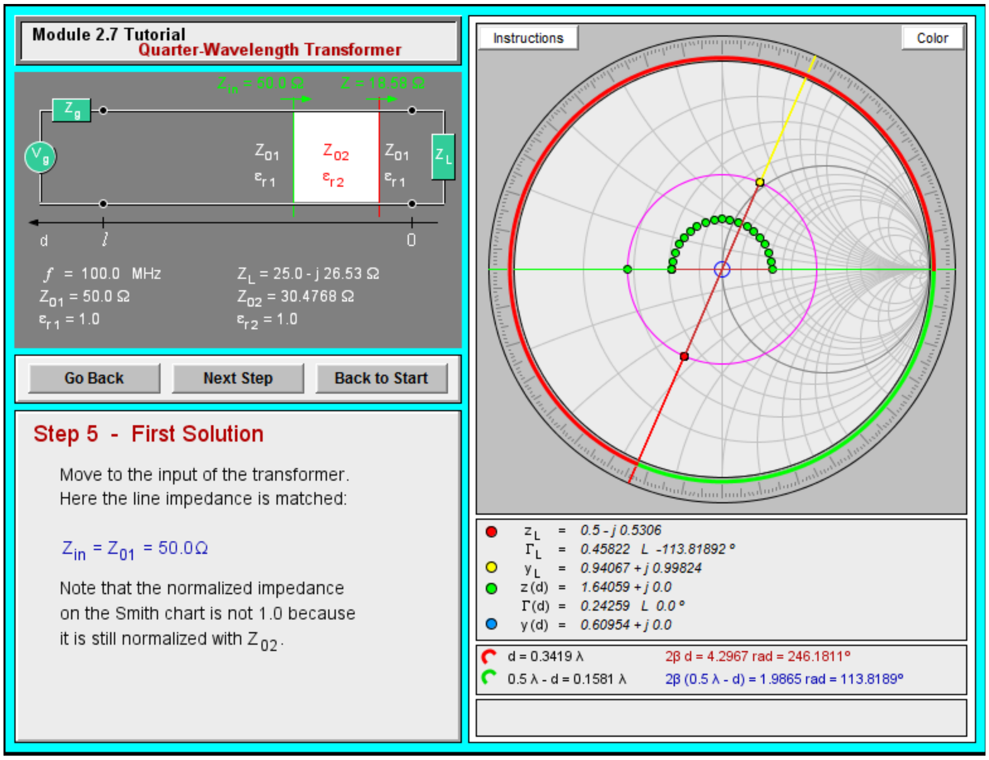


Figure 2.65: Quarter-wave match Smith chart: Step 5

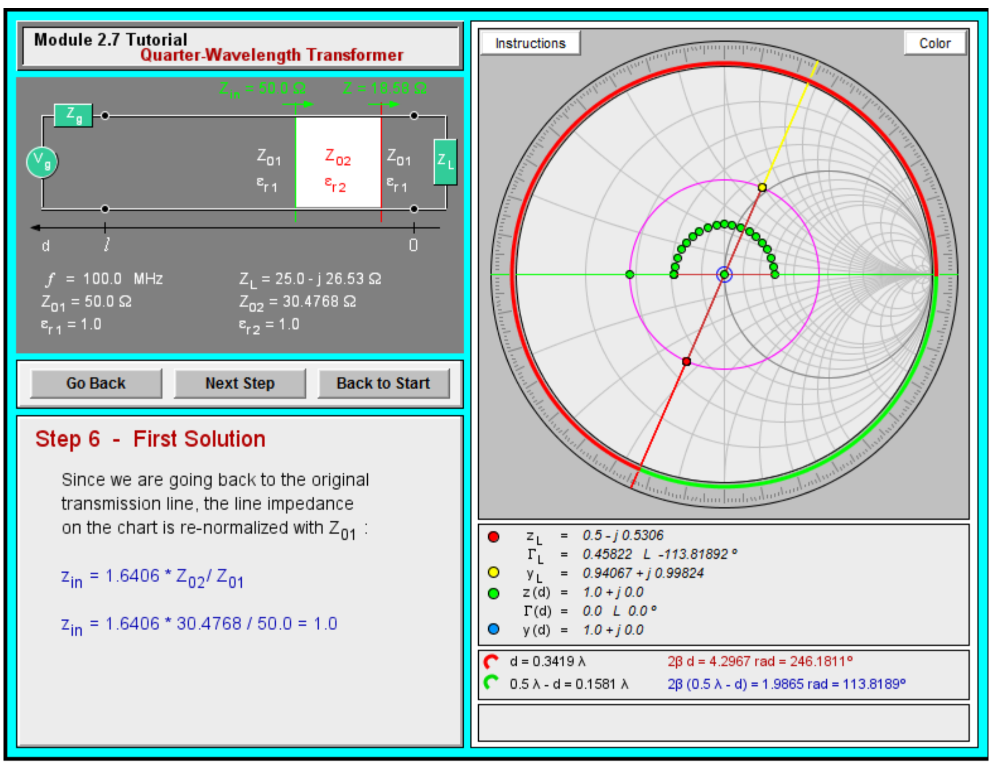


Figure 2.66: Quarter-wave match Smith chart: Step 6

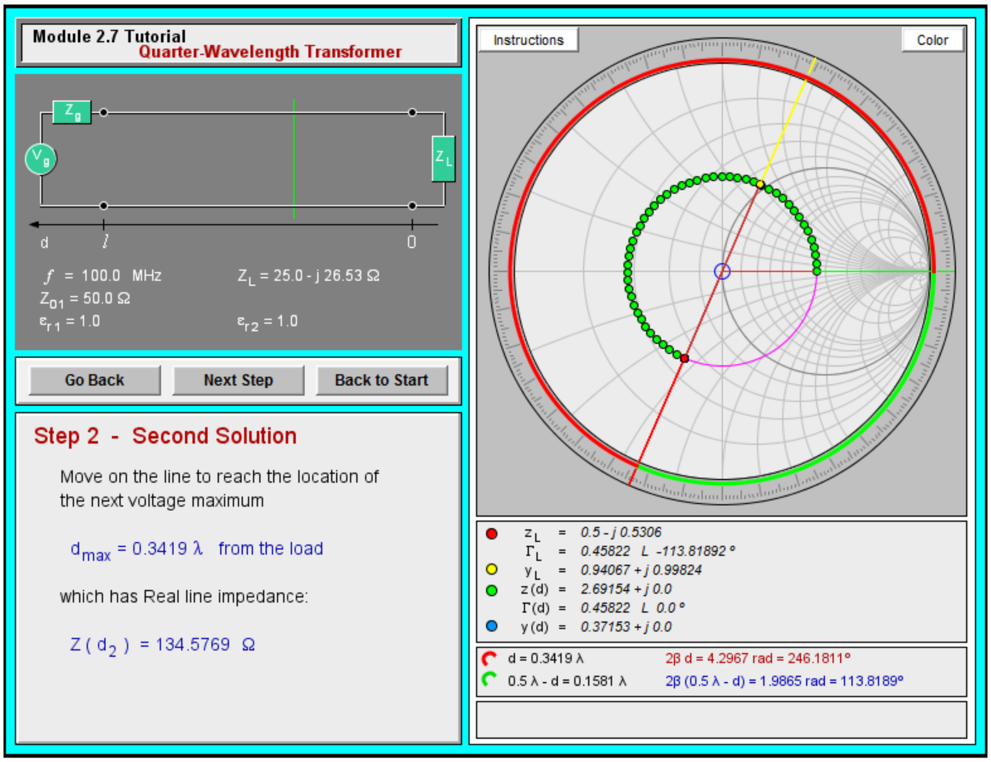


Figure 2.67: Quarter-wave match Smith chart: Solution 2 set-up

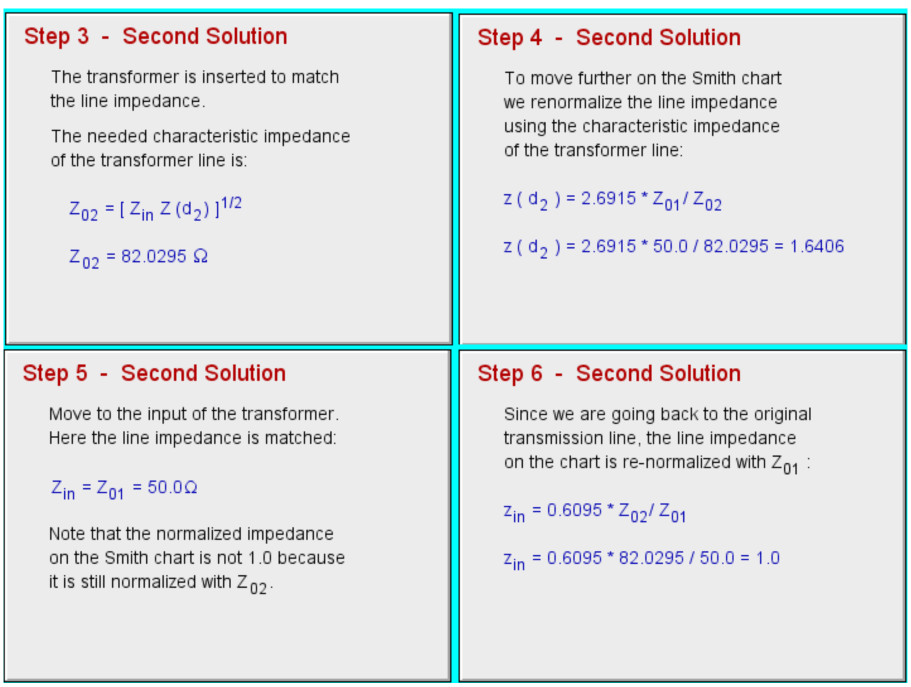


Figure 2.68: Quarter-wave match Smith chart: Solution 2

2.11.2 Lumped Element Matching

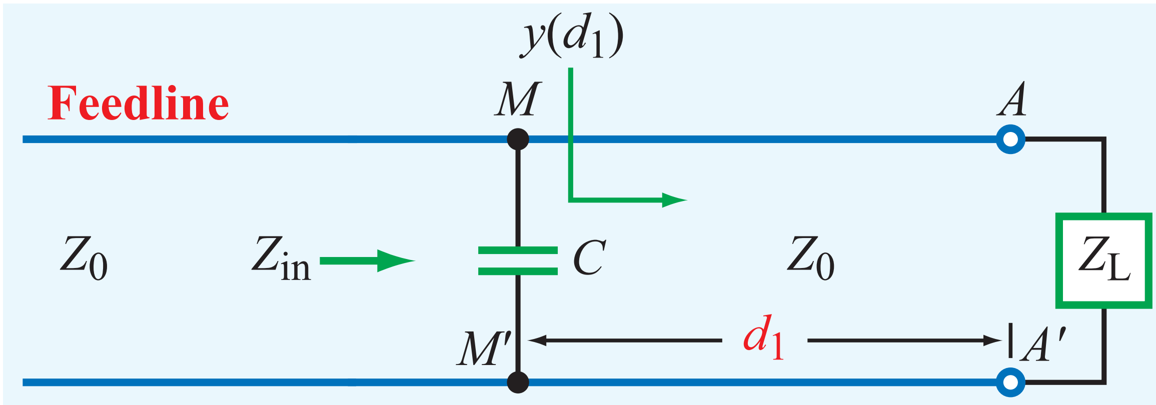


Figure 2.69: Lumped element matching using a shunt C

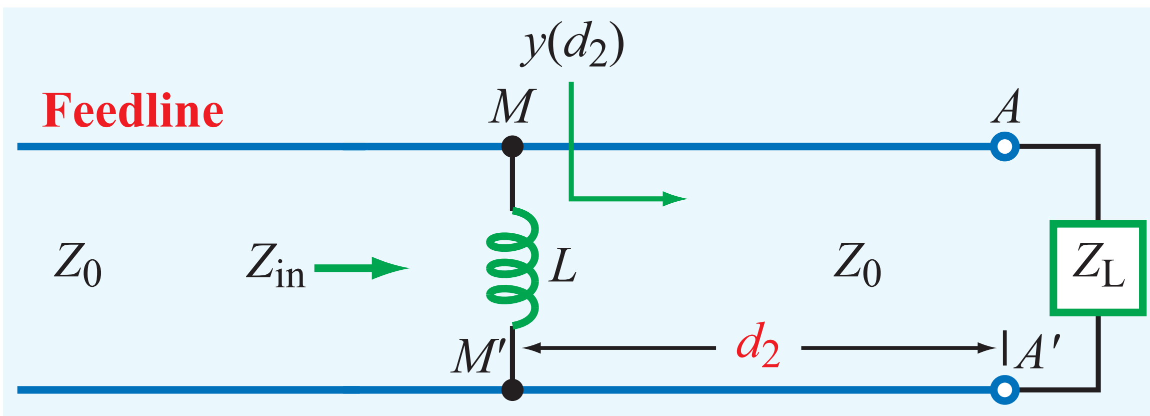


Figure 2.70: Lumped element matching using a shunt L

- In 2.50 circuits (c) and (d) consider a special case of lumped element matching, which employs a series transmission line connected to the load with a shunt L or C across the line

- As shunt elements are added across the through line, all of the analysis is done using admittances since admittances add in a parallel circuit, i.e., referring to either 2.69 or reffig:num37c

$$Z_{\text{in}} = \frac{1}{\frac{1}{Z_d} + \frac{1}{Z_s}}$$
$$Y_{\text{in}} = Y_d + Y_s$$

where Y_d is the admittance looking into the series line of length d_1 or d_2 and Y_s is the admittance of either a shunt C, i.e., $Y_C = j2\pi fC$, or a shunt L, i.e., $Y_s = 1/(j2\pi fL)$

- In practice $Y_d = G_d + jB_d$ is complex (a conductance and a susceptance), we will force to be real shortly, and Y_s is pure imaginary (inductor or capacitor), so $Y_s = jB_s$

- The objective is to force $Y_{\text{in}} = Y_0 + j0$ or in normalized form $y_{\text{in}} = 1 + j0$, meaning that

$$\boxed{\begin{aligned} g_d &= 1 \quad \text{real-part condition} \\ b_s &= -b_d \quad \text{imaginary-part condition} \end{aligned}}$$

where $g_d = g_D/Y_0$, $b_s = B_s/Y_0$, and $b_d = B_d/Y_0$

- The feedline is now matched at a single frequency

- The real-part condition is met by choosing the series line length d_1 or d_2 to make Y_{d_1} or Y_{d_2} respectively lie on the unit conductance circle of the Smith chart (this is always possible for a passive load)

- Once on the unit conductance circle, the remaining positive or negative susceptance is cancelled by the shunt L or C negative or positive susceptance respectively

- A fully analytic solution, is described in the text

- The key result of the analytical solution is that

$$\cos\left(\theta_r - 2\beta d\right) = -|\Gamma|,$$

which finds d and

$$b_s = \frac{2|\Gamma|\sin(\theta_r - 2\beta d)}{1 + |\Gamma|^2 + 2|\Gamma| + \cos\left(\theta_r - 2\beta d\right)}$$

which ultimately finds L or C

- For the upcoming example, the focus is on a Smith chart solution, as it is more straightforward and practical for working quiz and exam problems (not as accurate however)

Example 2.19: Series Line Shunt L or C Matching to a Series RC Load

- Once again the load is $R = 25\ \Omega$ in series with $C = 60$ pF operating at 100 MHz

- A Smith chart solution is formulated by following along with the tutorial of Module 2.8.

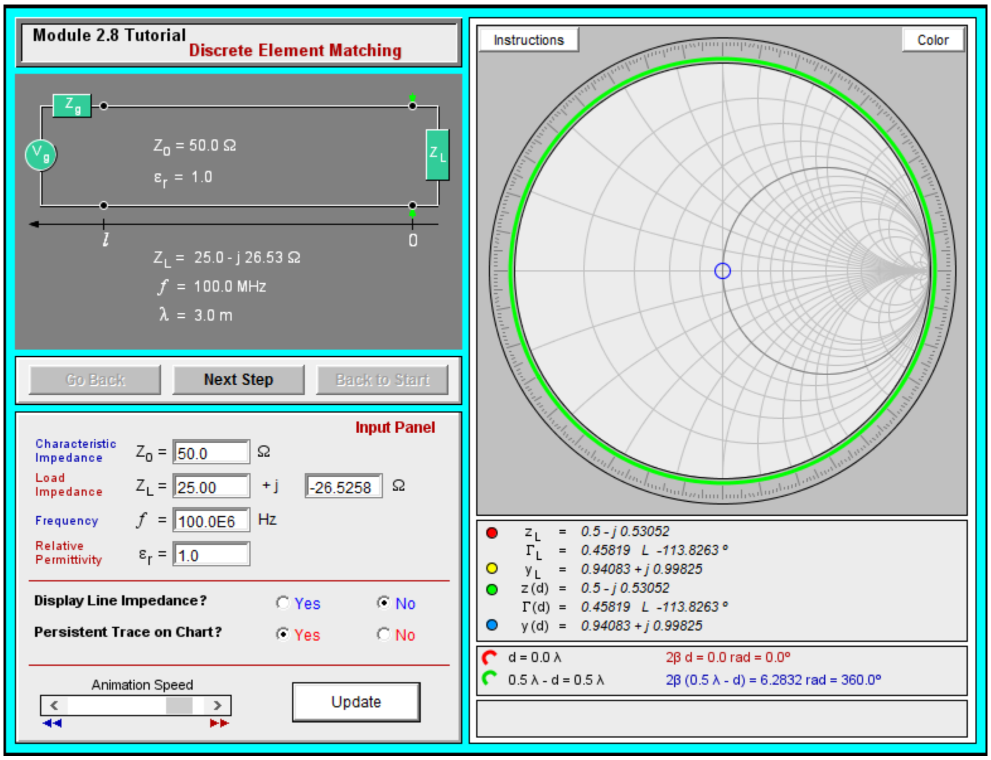


Figure 2.71: Lumped match Smith chart: Applet set-up

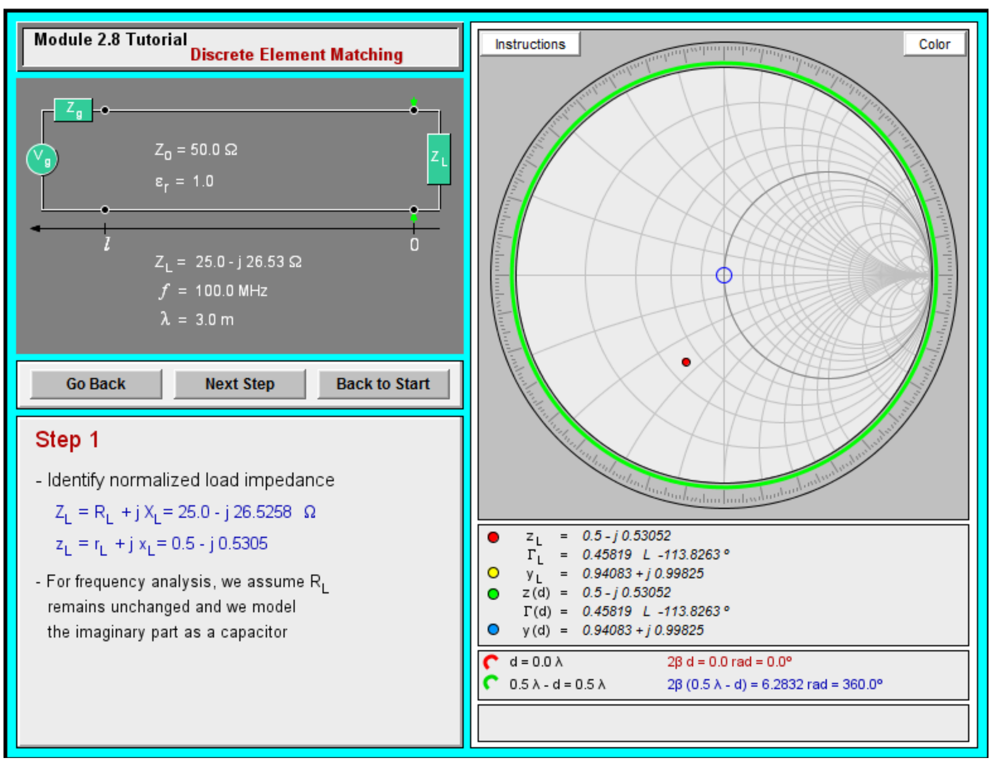


Figure 2.72: Lumped match Smith chart: Step 1

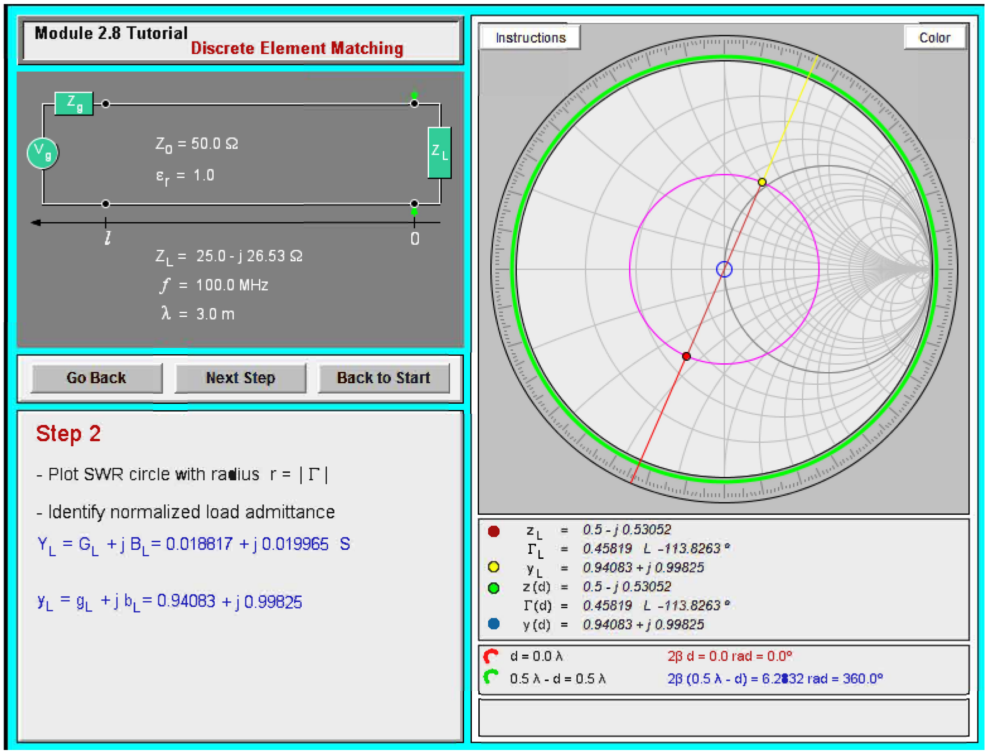


Figure 2.73: Lumped match Smith chart: Step 2

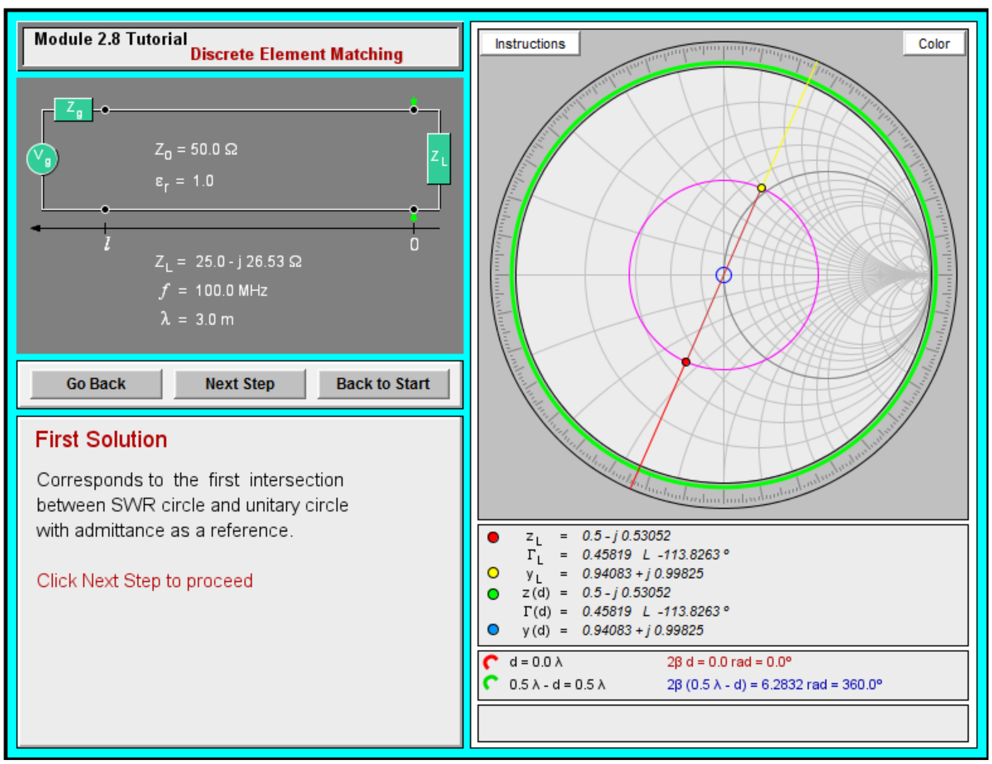


Figure 2.74: Lumped match Smith chart: First solution

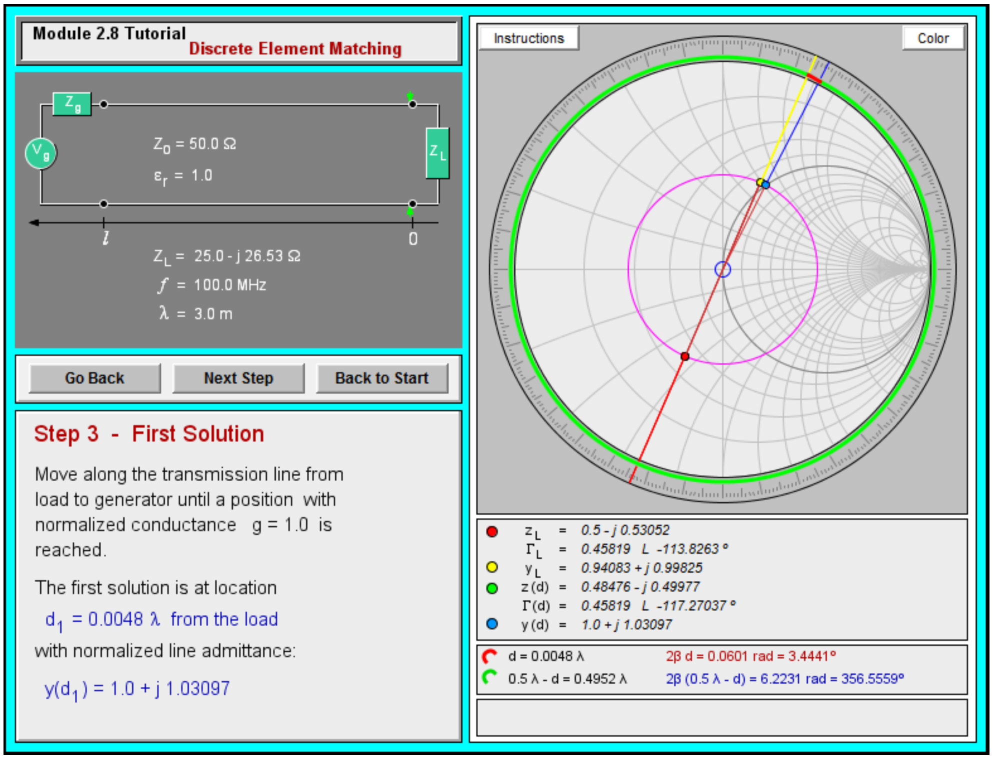


Figure 2.75: Lumped match Smith chart: Step 3

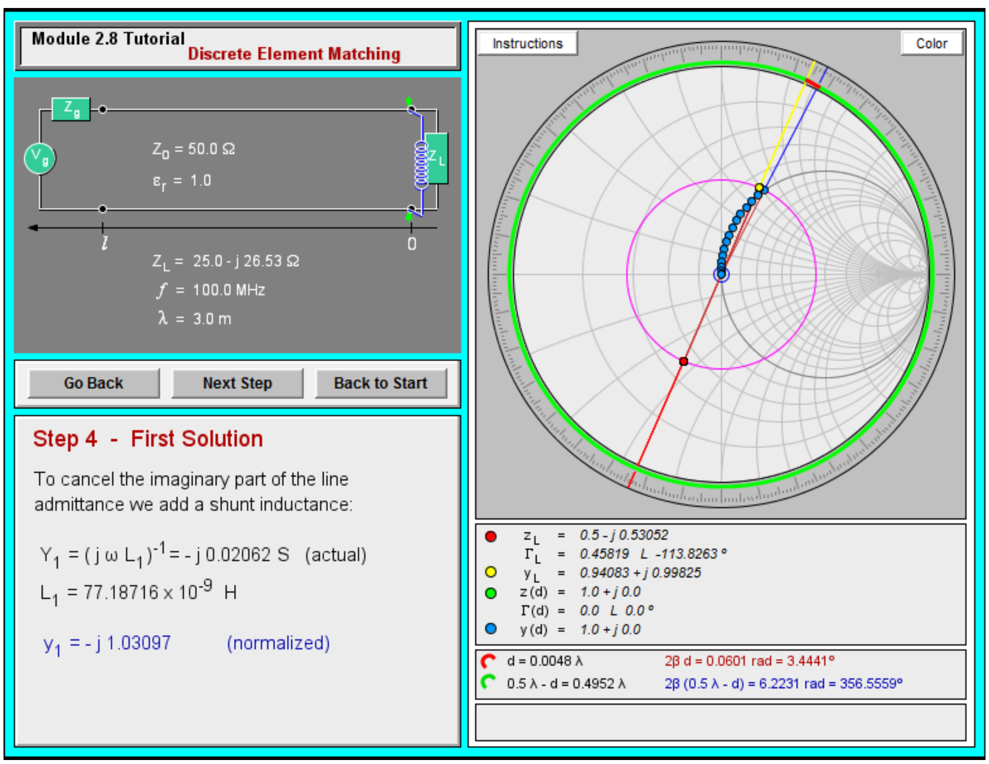


Figure 2.76: Lumped match Smith chart: Step 4

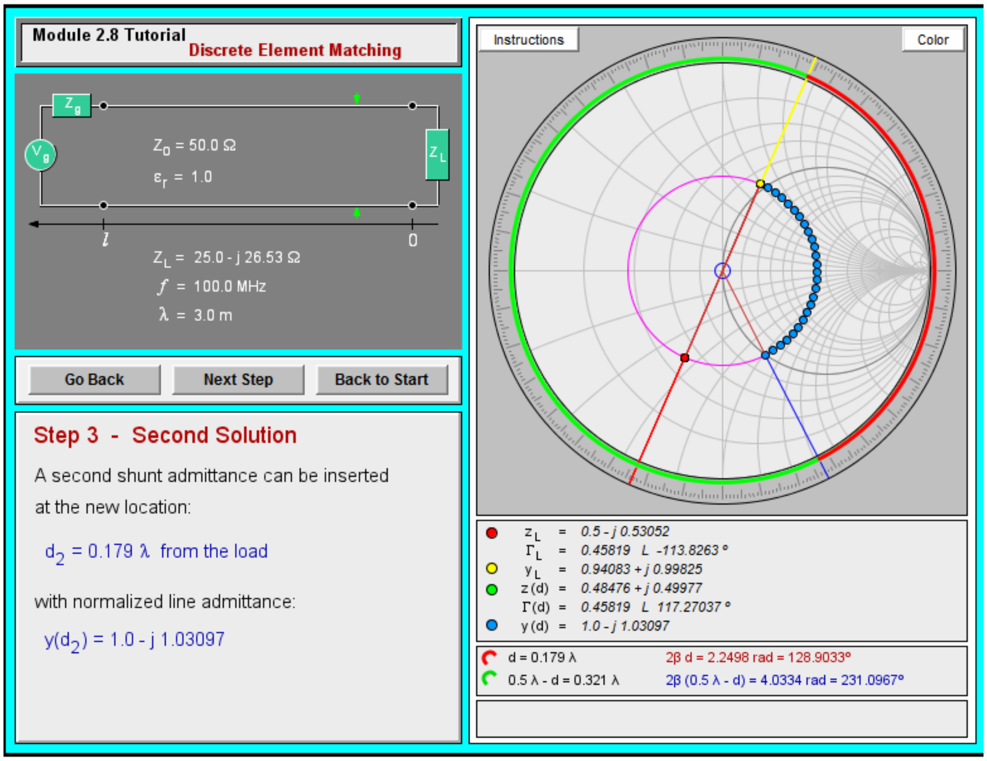


Figure 2.77: Lumped match Smith chart:Second solution Step 3

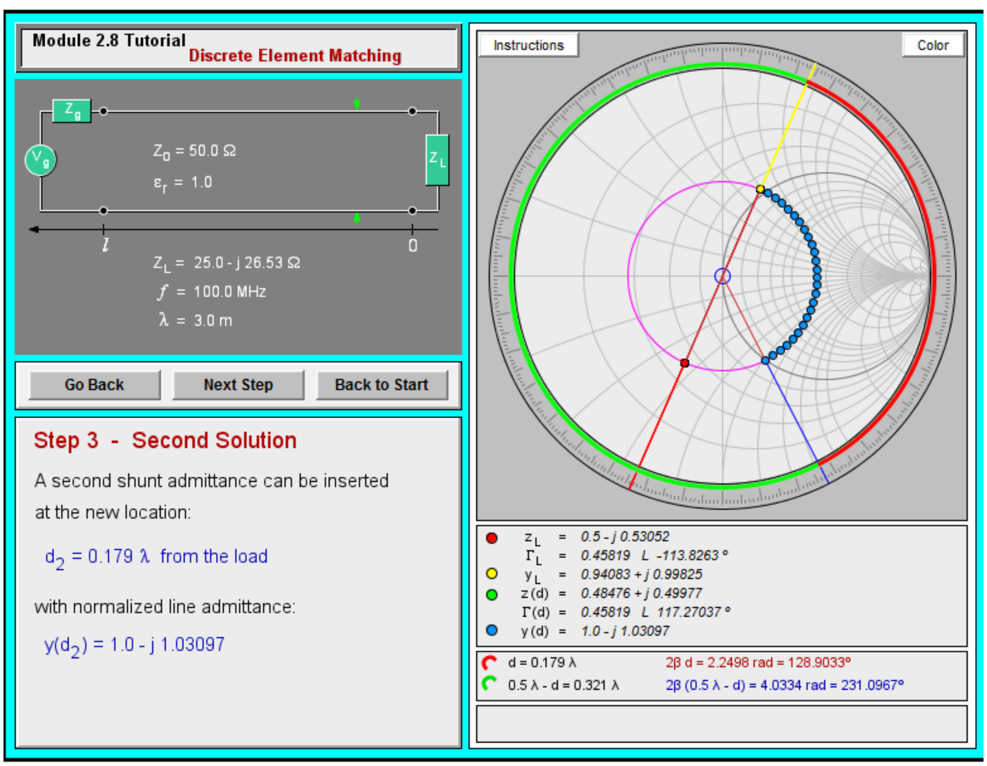


Figure 2.78: Lumped match Smith chart:Second solution Step 3

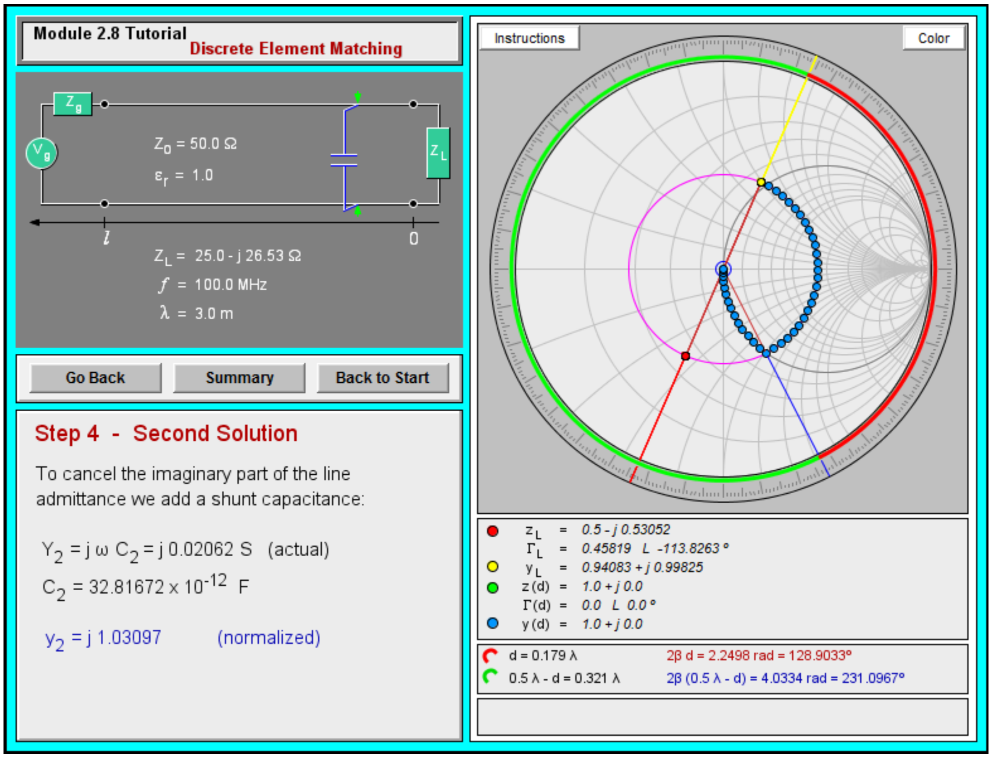


Figure 2.79: Lumped match Smith chart:Second solution Step 4

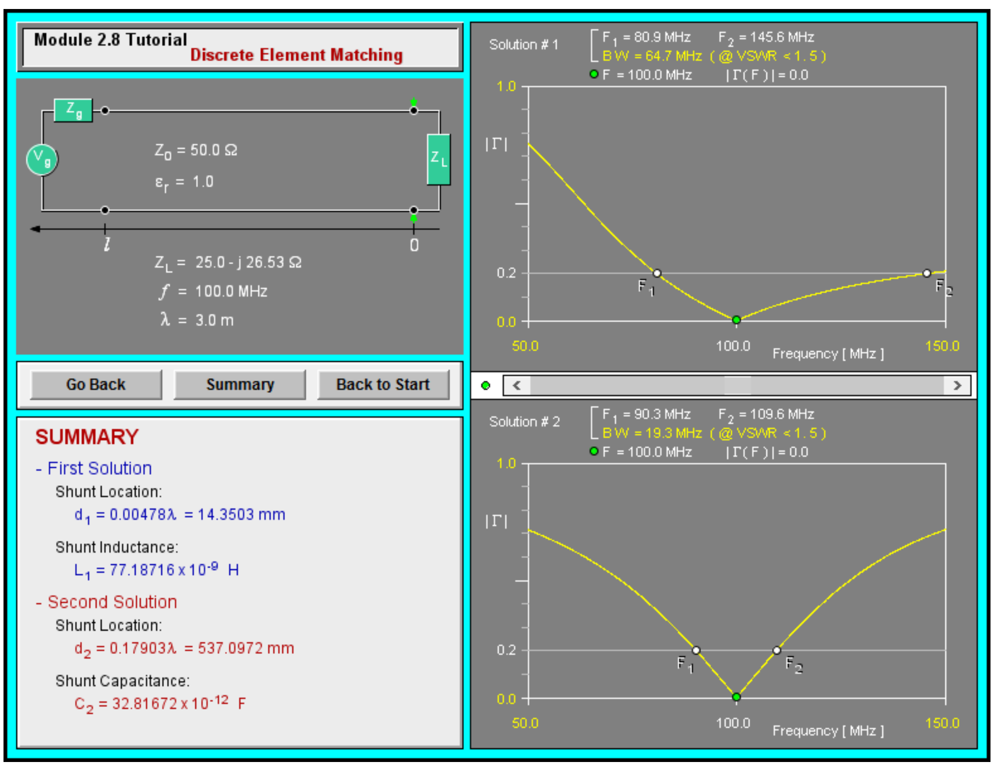


Figure 2.80: Lumped match Smith chart:Summary

- Remaining questions?

2.11.3 Single-Stub Matching

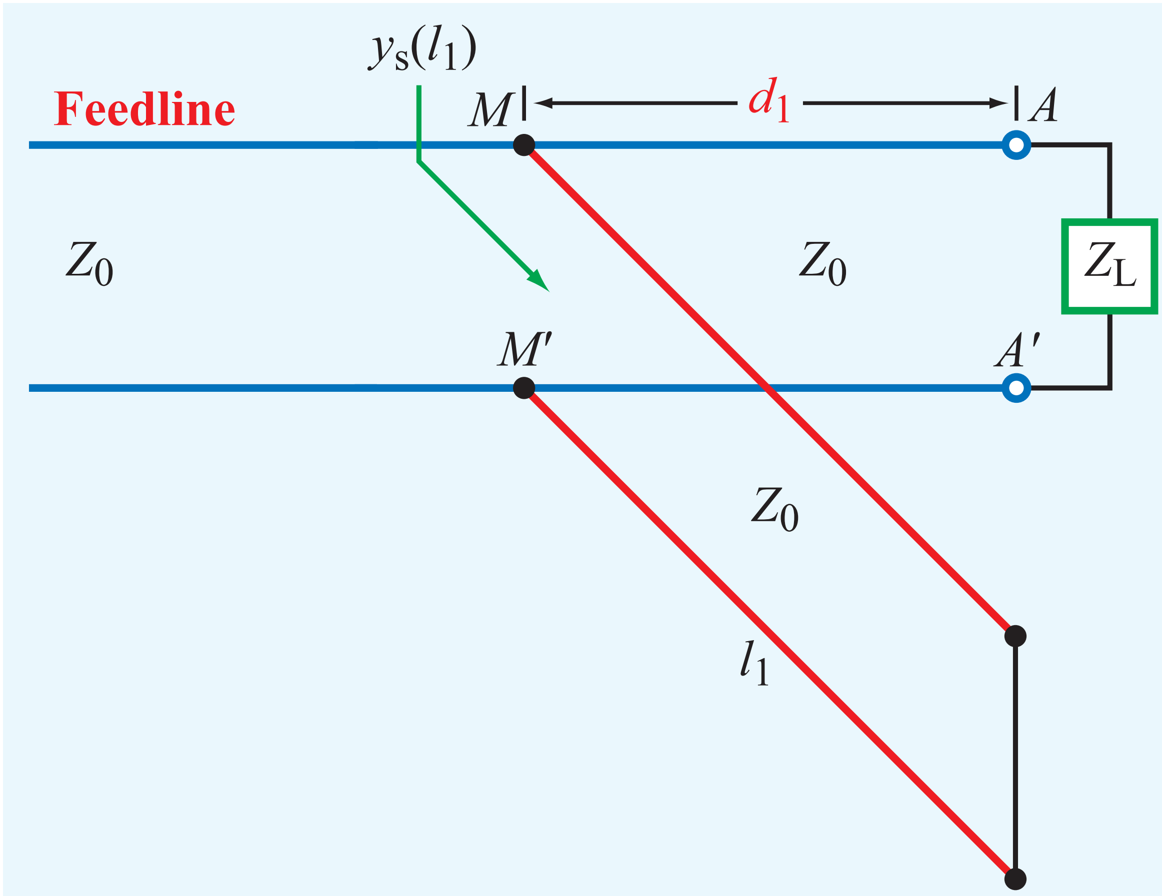


Figure 2.81: Stub matching using a shunt short circuit stub or an open-circuit stub (not shown).

- In 2.50 circuit (e) considers a special case of open/short circuit single stub matching, which employs a series transmission line connected to the load with a shunt open or short circuit stub transmission line shunted across the line
- The design steps are similar to the lumped element design presented earlier
- The main difference is the shunt L and C values are replaced with shunt open or short circuit stubs

  - For the case an open circuit stub the admittance is a pure susceptance $B_s = jY_0 \tan \beta l$ and the short circuit stub is also a pure susceptance $B_s = -jY_0 \cot \beta l$

-

- The tutorial of text Module 2.9 does a great job of explaining the steps

Example 2.20: Series Line Single Stub Matching to a Series RC Load

- Once again the load is $R = 25\ \Omega$ in series with $C = 60$ pF operating at 100 MHz
- A Smith chart solution is formulated by following along with the tutorial of Module 2.9.
- One item missing is?

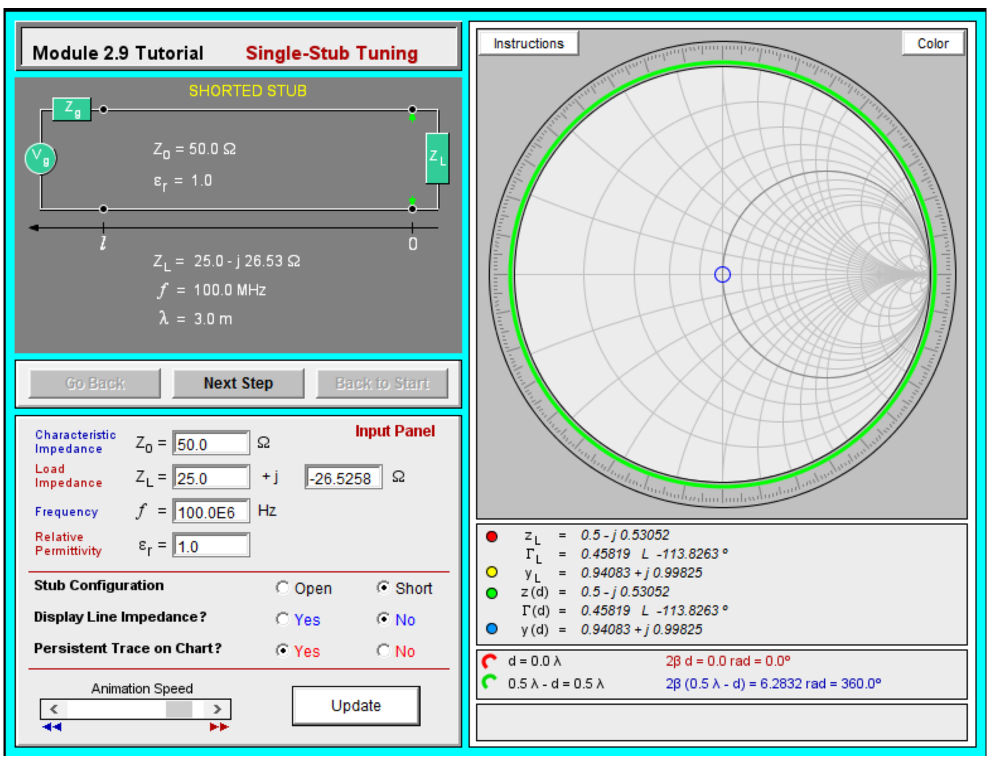


Figure 2.82: Stub match Smith chart: Applet set-up

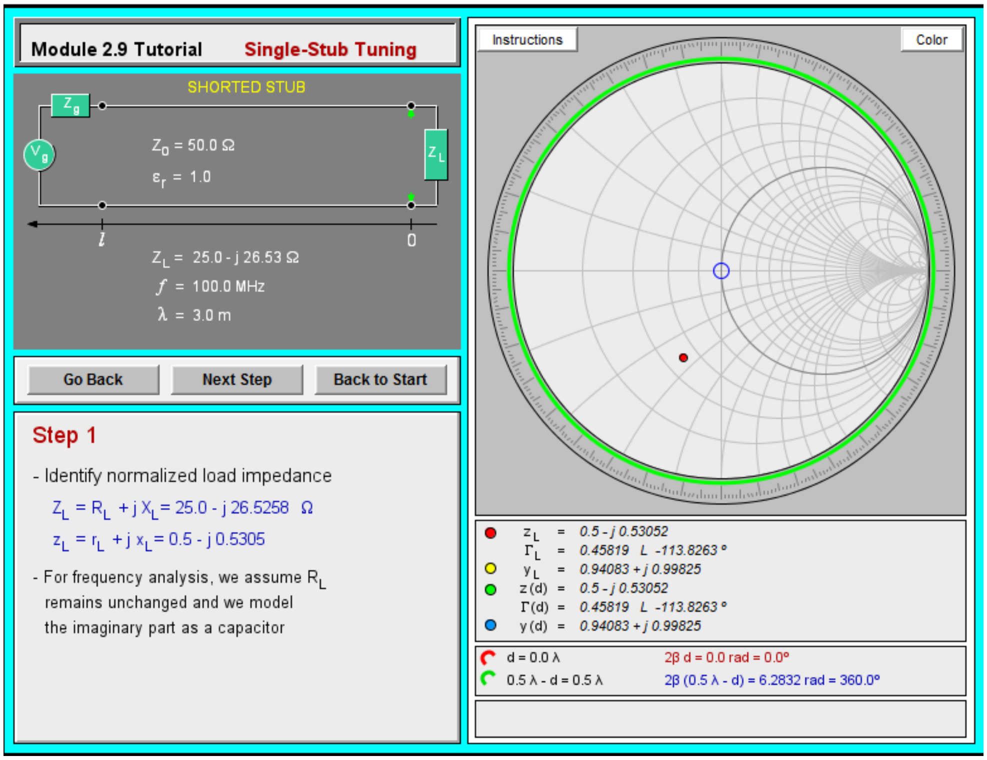


Figure 2.83: Stub match Smith chart: Step 1

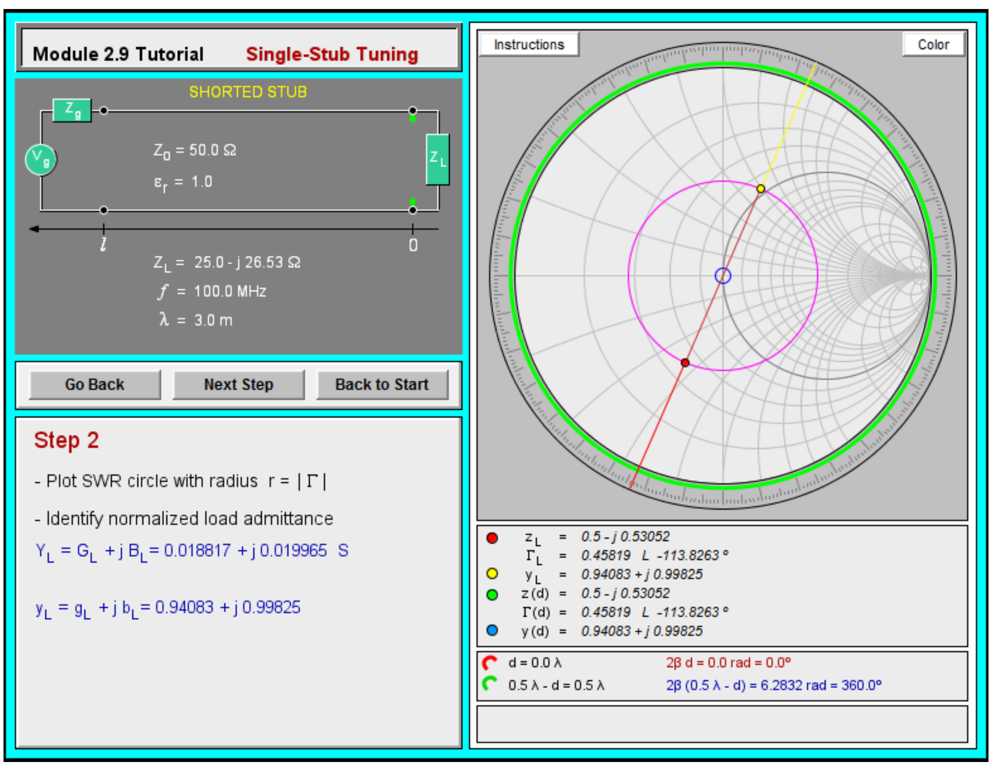


Figure 2.84: Stub match Smith chart: Step 2

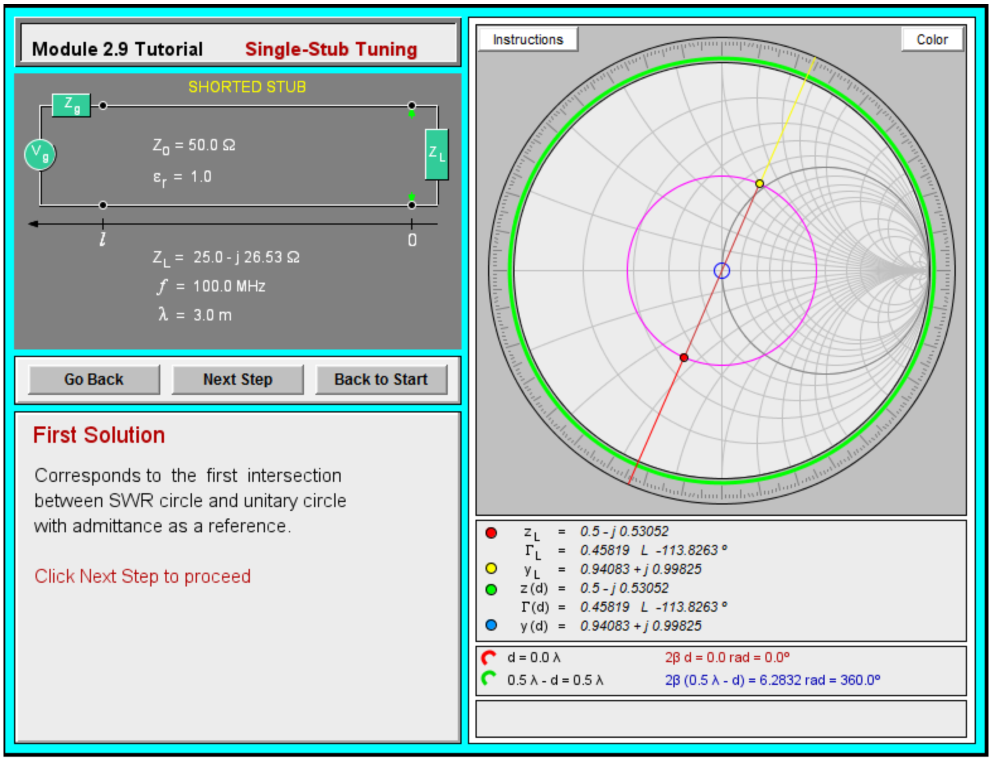


Figure 2.85: Stub match Smith chart: First Solution

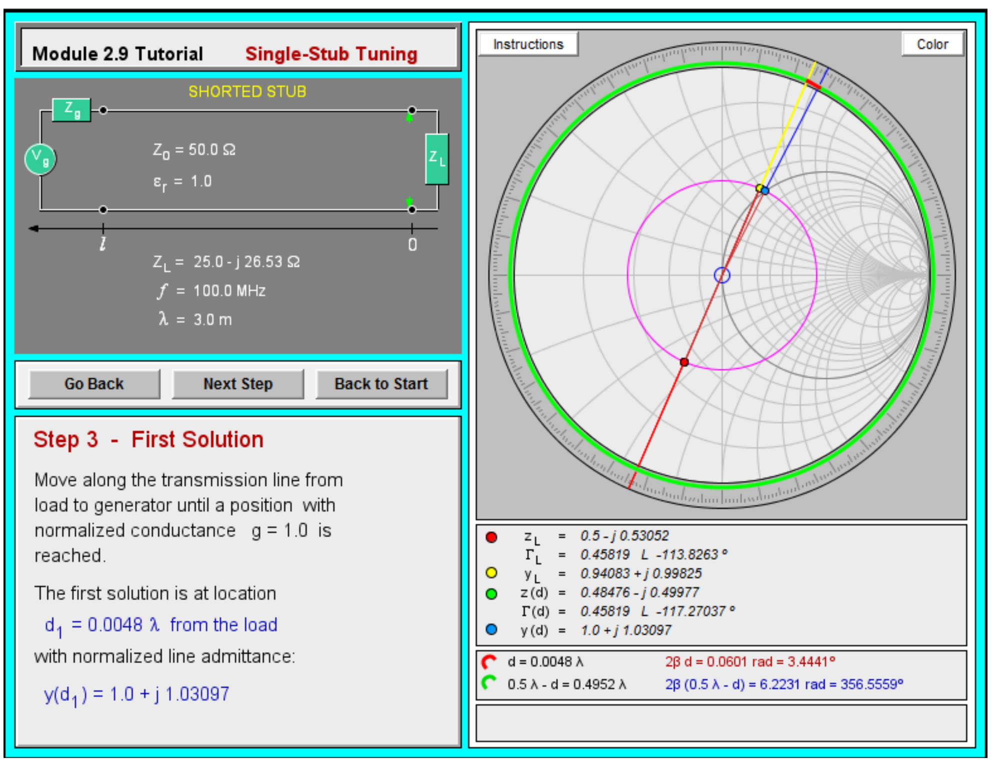


Figure 2.86: Stub match Smith chart: Step 3

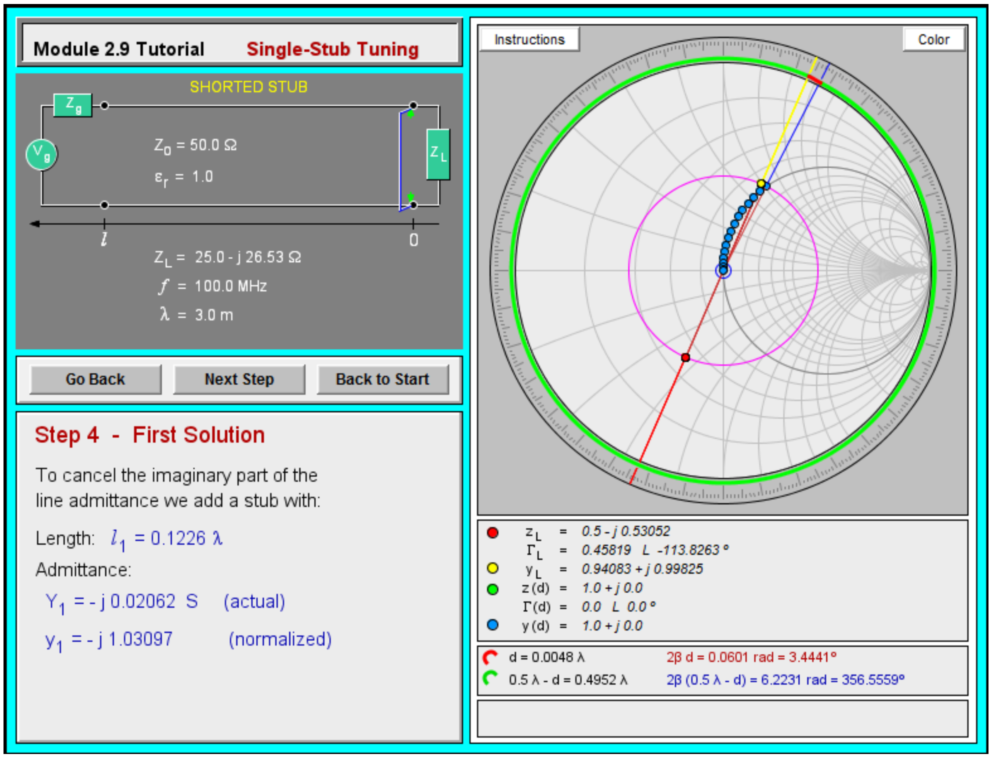


Figure 2.87: Stub match Smith chart: Step 4

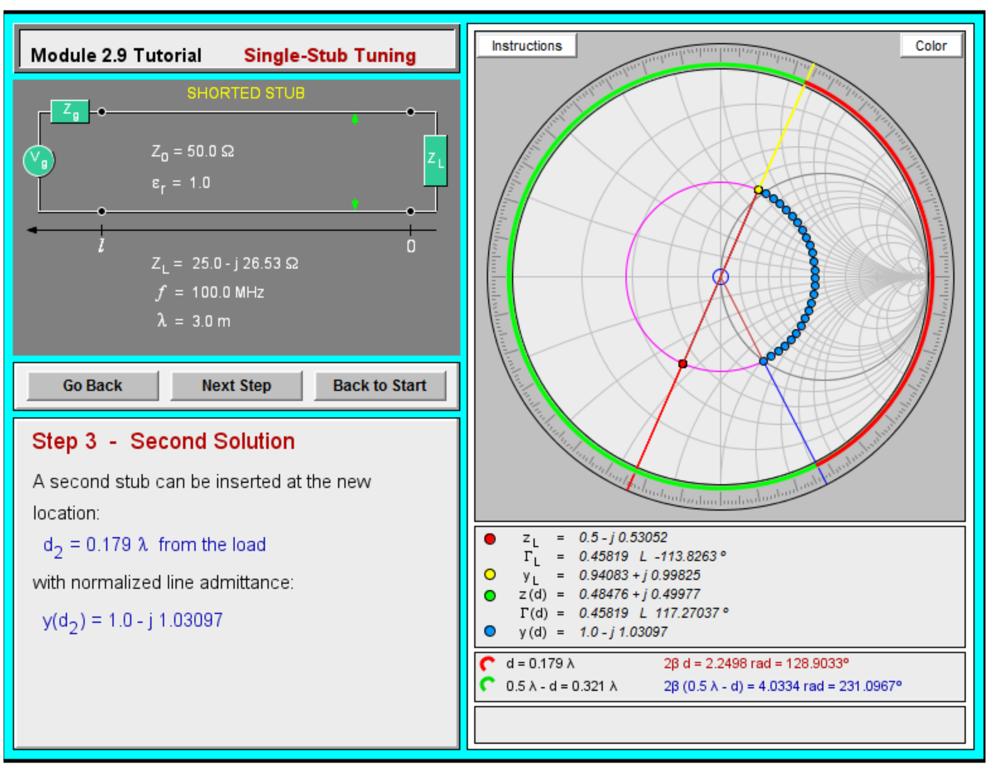


Figure 2.88: Stub match Smith chart: Solution 2 Step 3

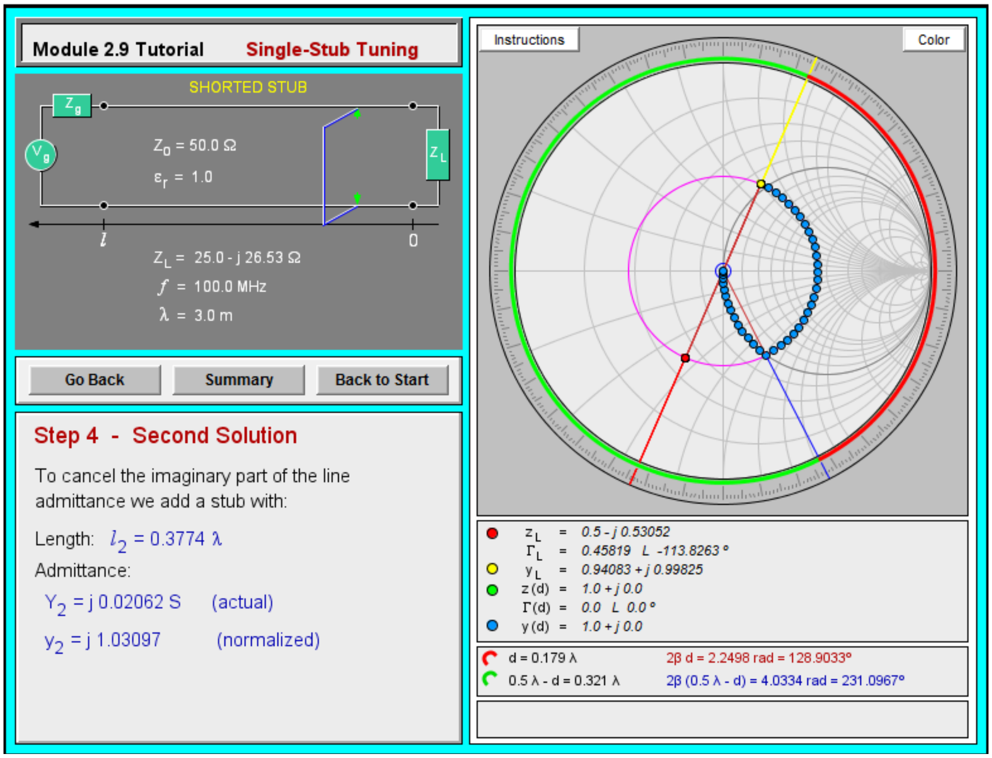


Figure 2.89: Stub match Smith chart: Solution 2 Step 4

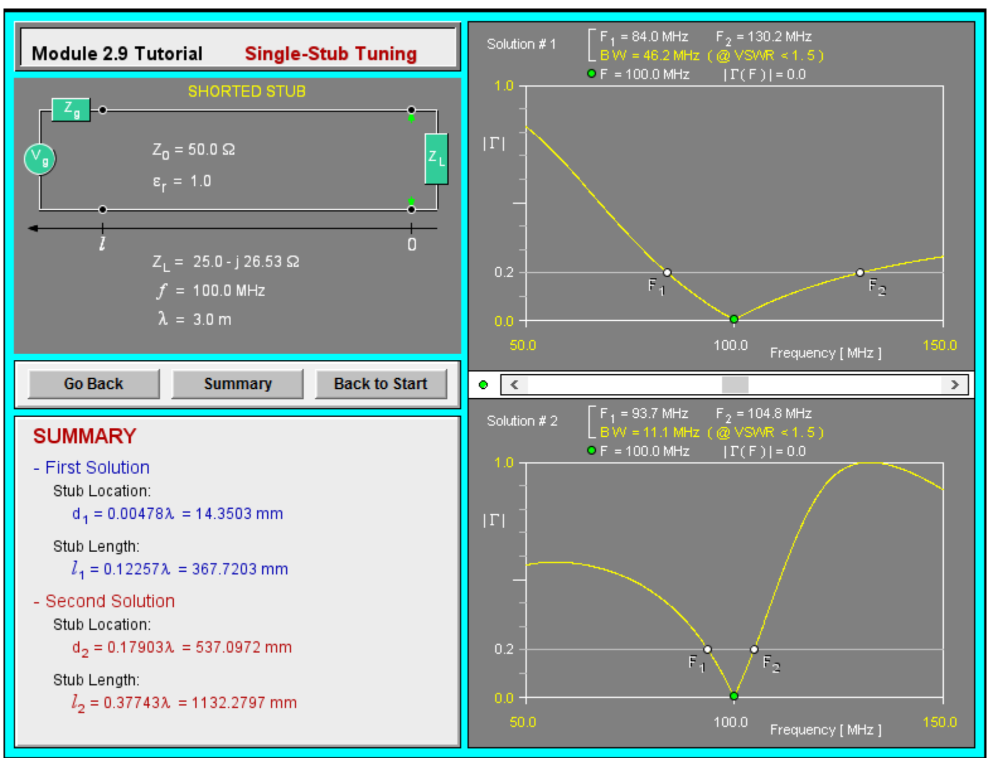


Figure 2.90: Stub match Smith chart: Summary

2.12 Transients on Transmission Lines

- Transmission lines carry more than pure sinusoidal signals
- Many applications of transmission lines do involve sinusoidal-based signals, such communications waveforms where the amplitude, frequency, or phase of a sinusoid is a made a function of a *message* signal, analog or digital
- The high frequency microwave/radio frequency sinsuoid serves as a *carrier* of the information
- Still, the spectral bandwidth of these *modulated carrier* waveforms is small relative to the carrier frequency
- True *wideband* signals or so-called *baseband* digital waveforms are clocked at 100 MHz and above, e.g., the PCB tracks forming a data bus on PC motherboard or the data bus of a flexible PCB on an array of disk drives in storage rack
- This is where time-domain of transmission lines comes to the stage
- Time domain modeling may even resort to Laplace transform methods and will make use of simulations using LTspice in the `.tran`
- Two useful signals are the unit step function $u(t)$ and a rectangular pulse of duration τ s
  - A step signal truning on at $t = 0$ takes the form

$$V(t) = V_0 u(t)$$

– A pulse of duration τ s that turns on at $t = 0$ takes the form

$$V(t) = V_0 u(t) - V_0 u(t - \tau)$$

where in both cases $u(t)$ is the unit step function defined as

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

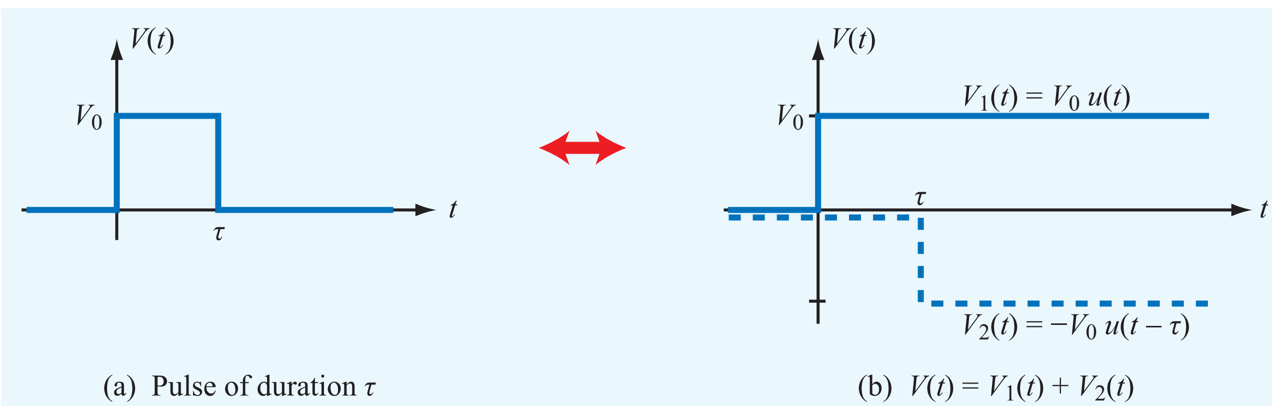


Figure 2.91: Forming a pulse signal as the differnce between two step functions skewed in turn-on time by τ s.

2.12.1 Response to a Step Function

- To get started with transient modeling, consider a single section of transmission line with resistive load and source terminations

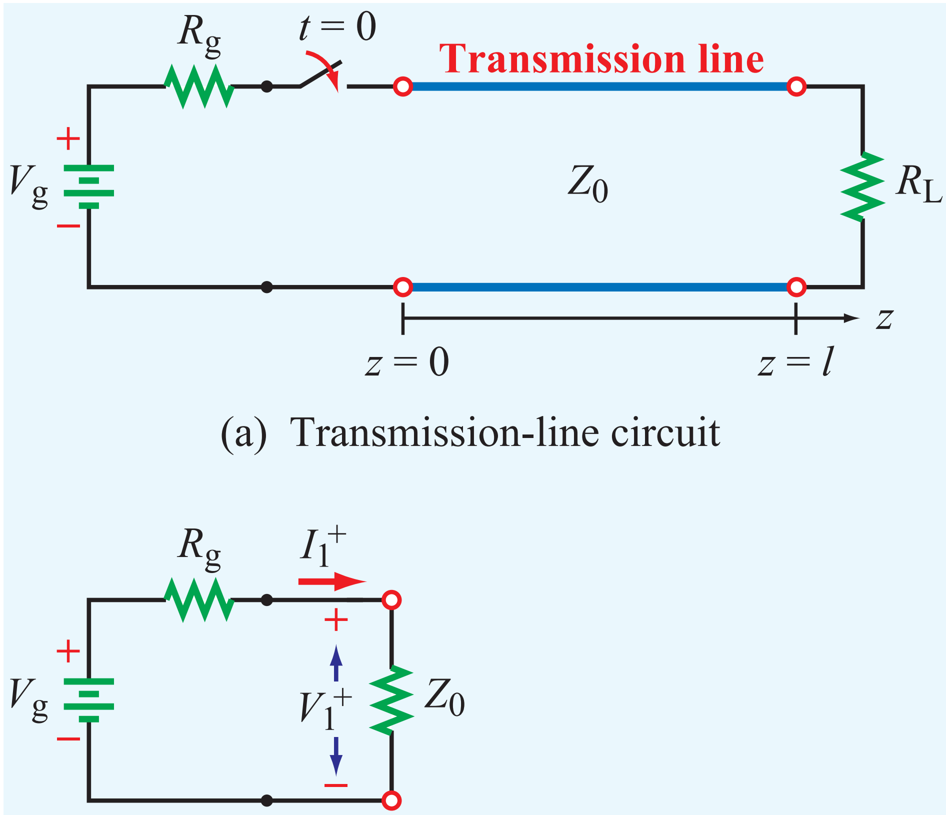


Figure 2.92: Circuit model for transient analysis.

- **Big change**: The source end is now taken as $z = 0$ and the load end is $z = l$

- Upon closing the switch (step function $u(t)$ turns on) incident voltage, V_1^+, and current, I_1^+, waves are launched on the line

- The boundary conditions dictate that only the line impedance Z_0 is all that is seen by the generator at $t = 0$, so

$$I_1^+ = \frac{V_g}{R_g + Z_0}$$
$$V_1^+ = \frac{V_g Z_0}{R_g Z_0}$$

- The subscript 1 on V^+ and I^+ denotes the initial voltage and current wave launched onto the line in the $+z$ direction

2.12.2 Dr. Wickert's Archive Notes

- At this point we jump to some pages taken from the intro chapter to ECE 4250/5250
- The title of this chapter is *Transmission Theory Review*, so don't be too worried about the course number

Review of Transmission Line Theory

Chapter **1**

- Transmission lines and waveguides are used to transport electromagnetic energy at microwave frequencies from one point in a system to another
- The desirable features of a transmission line or waveguide are:
  - Single-mode propagation over a wide band of frequencies
  - Small attenuation
- The transmission line structures of primary interest for this course are those for which the dominant mode of propagation is a transverse electromagnetic (TEM) wave
- Recall that for TEM waves the components of electric and magnetic fields in the direction of wave propagation are zero
- We wish to consider transmission lines which consist of two or more parallel conductors which have axial uniformity
- That is to say their cross-sectional shape and electrical properties do not vary along the axis of propagation

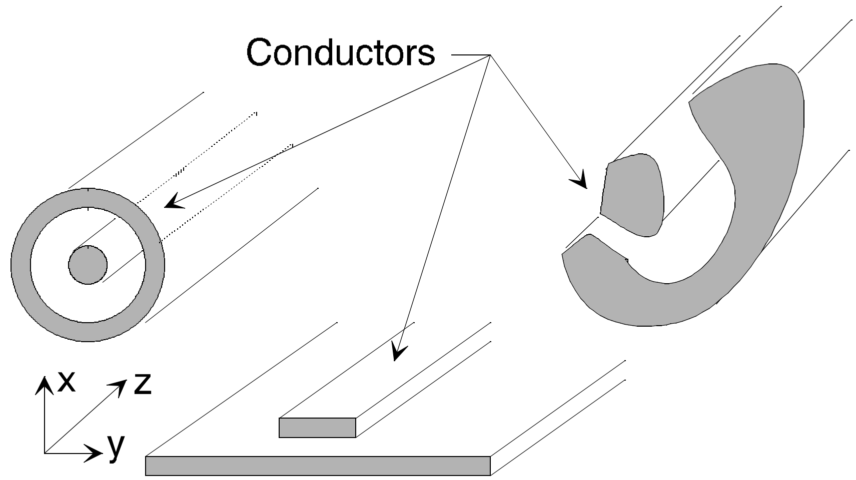


Figure 1.1: Transmission lines with axial uniformity

- The TEM wave solution for axially uniform transmission lines can be obtained using:
  - Field Analysis - (obtain electric and magnetic field waves analogous to uniform plane waves)
  - Distributed-Circuit Analysis - (obtain voltage and current waves)
- Distributed circuit analysis will be at the forefront of all analysis in this course, in particular consider Pozar[1], "Modern microwave engineering involves predominantly distributed circuit analysis and design, in contrast to the waveguide and field theory orientation of earlier generations"

1. David Pozar, *Microwave Engineering*, 3rd edition, John Wiley, New York, 2005.

Distributed Circuit Analysis

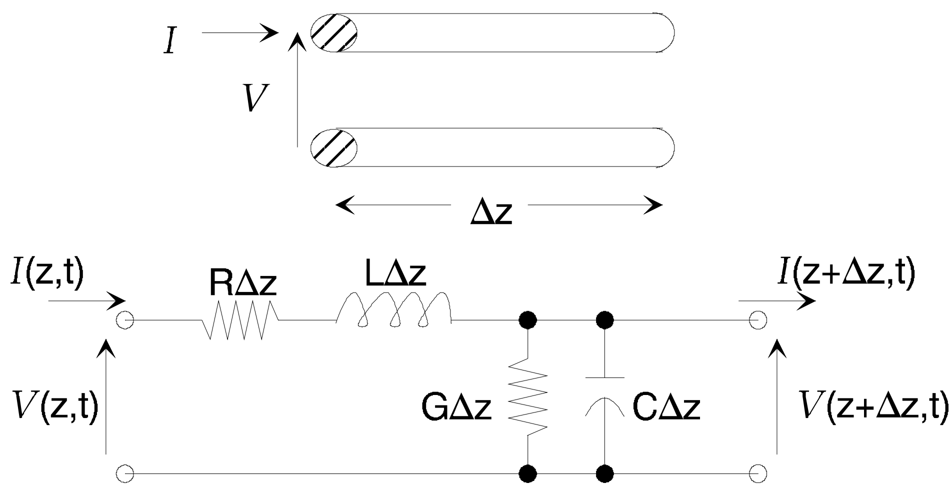


Figure 1.2: Distributed element circuit model

Parameters:

L – series inductance per unit length due to energy storage in the magnetic field

C – shunt capacitance per unit length due to energy storage in the electric field

R – series resistance per unit length due to power loss in the conductors

G – shunt conductance per unit length due to power loss in the dielectric. (i.e. $\varepsilon = \varepsilon' - j\varepsilon''$, $\varepsilon'' \neq 0$)

- Using KCL, KVL and letting $\Delta z \to 0$ it can be shown that

$$\frac{-\partial v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t} \tag{1.1}$$

and

$$\frac{-\partial i(z,t)}{\partial z} = Gv(z,t) + C\frac{\partial v(z,t)}{\partial t} \tag{1.2}$$

- For now assume the line is lossless, that is $R = 0$ and $G = 0$, so we have:

$$
\begin{aligned}
-\frac{\partial v}{\partial z} &= L\frac{\partial i}{\partial t} \\
-\frac{\partial i}{\partial z} &= C\frac{\partial v}{\partial t}
\end{aligned}
\tag{1.3}
$$

- Now differentiate the first equation with respect to z and the second with respect to time t

$$
\begin{aligned}
\frac{\partial^2 v}{\partial z^2} &= -L\frac{\partial^2 i}{\partial t \partial z} \\
\frac{\partial^2 i}{\partial z \partial t} &= C\frac{\partial^2 v}{\partial t^2}
\end{aligned}
\tag{1.4}
$$

- Combine the two resulting equations to get

$$\frac{\partial^2 v}{\partial z^2} = LC\frac{\partial^2 v}{\partial t^2} \text{ (voltage eqn.)} \tag{1.5}$$

similarly obtain

$$\frac{\partial^2 i}{\partial z^2} = LC\frac{\partial^2 i}{\partial t^2} \text{ (current eqn.)} \tag{1.6}$$

- These are in the form of the classical one-dimensional wave equation, often seen in the form

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{\nu_p^2}\frac{\partial^2 y}{\partial t^2} \tag{1.7}$$

where ν_p has dimension and significance of velocity

- A well known solution to the wave equation is

$$y = y^{+}\left(t - \frac{z}{\nu_p}\right) + y^{-}\left(t + \frac{z}{\nu_p}\right) \tag{1.8}$$

  – y^+ propagates in the positive z direction

  – y^- propagates in the negative z direction

- This solution can be checked by noting that

$$\frac{\partial y^{\pm}}{\partial z} = \frac{\partial y^{\pm}}{\partial x}\frac{\partial x}{\partial z} = \frac{\mp 1}{\nu_p}\frac{\partial y^{\pm}}{\partial x}, x = t \pm \frac{z}{\nu_p} \tag{1.9}$$

- A solution for the voltage wave equation is thus

$$v(z, t) = v^{+}\left(t - \frac{z}{\nu_p}\right) + v^{-}\left(t + \frac{z}{\nu_p}\right) \tag{1.10}$$

- The current equation can be written in a similar manner, but it can also be written in terms of V^+ and V^- since

$$\frac{-\partial v}{\partial z} = L\frac{\partial i}{\partial t} \tag{1.11}$$

  or

$$-\left[\frac{\partial v^{+}}{\partial z} + \frac{\partial v^{-}}{\partial z}\right] = L\frac{\partial i}{\partial t} \tag{1.12}$$

  Letting $x = t \pm z/\nu_p$ and using the chain rule the above equation becomes

$$-\left[\frac{-1}{\nu_p}\frac{\partial v^{+}}{\partial x} + \frac{1}{\nu_p}\frac{\partial v^{-}}{\partial x}\right] = L\frac{\partial i}{\partial t} \tag{1.13}$$

- This implies that

$$i(z, t) = \frac{1}{L\nu_p}\left[v^+\left(t - \frac{z}{\nu_p}\right) - v^-\left(t + \frac{z}{\nu_p}\right)\right] \tag{1.14}$$

or

$$i(z, t) = \frac{1}{Z_0}\left[v^+\left(t - \frac{z}{\nu_p}\right) - v^-\left(t + \frac{z}{\nu_p}\right)\right] \tag{1.15}$$

where

$$\begin{aligned} \nu_p &= \frac{1}{\sqrt{LC}}, \text{ (velocity of propagation)} \\ Z_0 &= \sqrt{\frac{L}{C}}, \text{ (characteristic impedance)} \end{aligned} \tag{1.16}$$

- At this point the general lossless line solution is incomplete. The functions v_+ and v_- are unknown, but must satisfy the boundary conditions imposed by a specific problem
- The time domain solution for a lossless line, in particular the analysis of transients, can most effectively be handled by using Laplace transforms
- If the source and load impedances are pure resistances and the source voltage consists of step functions or rectangular pulses, then time domain analysis is most convenient
- In the following we will first consider resistive load and source impedances, later the analysis will be extended to complex impedance loads using Laplace transforms

- The Laplace transform technique will in theory allow for a generalized time-domain analysis of transmission lines
- In Section 2 sinusoidal steady-state analysis will be introduced. This approach offers greatly reduced analysis complexity

Transient Analysis with Resistive Loads

Infinite Length Line

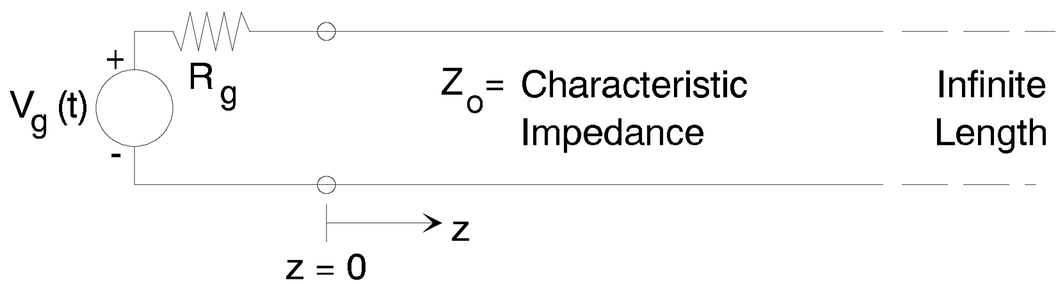


Figure 1.3: Infinite length line circuit diagram

- Assume that the line is initially uncharged, i.e. for $z \geq 0$ and $t \leq 0$

$$v(z, t) = v^{+}\left(t - \frac{z}{\nu_p}\right) + v^{-}\left(t + \frac{z}{\nu_p}\right) = 0 \tag{1.17}$$

and

$$i(z, t) = \frac{1}{Z_0}\left[v^{+}\left(t - \frac{z}{\nu_p}\right) - v^{-}\left(t + \frac{z}{\nu_p}\right)\right] = 0 \tag{1.18}$$

the above equations imply that

$$v^{+}\left(t - \frac{z}{\nu_p}\right) = 0 \text{ for } t - z/\nu_p < 0 \tag{1.19}$$

and

$$v^{-}\left(t+\frac{z}{\nu_p}\right) = 0 \ \text{ for all } t \tag{1.20}$$

Note: For the given initial conditions only $v^{+}(t - z/\nu_p)$ can exist on the line.

- We thus conclude that

$$\left.\begin{aligned} v(z,t) &= v^{+}\left(t-\frac{z}{\nu_p}\right) \\ i(z,t) &= \frac{1}{Z_0}v^{+}\left(t-\frac{z}{\nu_p}\right) \end{aligned}\right\} \text{for all } t-\frac{z}{\nu_p}>0 \tag{1.21}$$

- Suppose that at $t = 0^{+}$ a voltage source $v_g(t)$ is applied through a source resistance R_g, at $z = 0$
- Apply Ohm's law at $z = 0$ for $t > 0$ and we obtain

$$v_g(t) - v(0,t) = i(0,t)R_g \tag{1.22}$$

or

$$v_g(t) - v^{+}(t) = \frac{R_g}{Z_g}v^{+}(t) \tag{1.23}$$

which implies

$$v^{+}(t) = \frac{Z_0}{Z_0 + R_g}v_g(t) \tag{1.24}$$

- The final result is that under the infinite line length assumption for any z we can write

$$v(z, t) = \frac{Z_0}{Z_0 + R_g} v_g\left(t - \frac{z}{\nu_p}\right) \tag{1.25}$$

$$i(t, z) = \frac{1}{Z_0 + R_g} v_g\left(t - \frac{z}{\nu_p}\right) \tag{1.26}$$

  – **Note**: That the infinite length of line appears as a voltage divider to the source

  – Voltages and currents along the line appear as replicas of the input values except for the time delay z/ν_p

Terminated Line

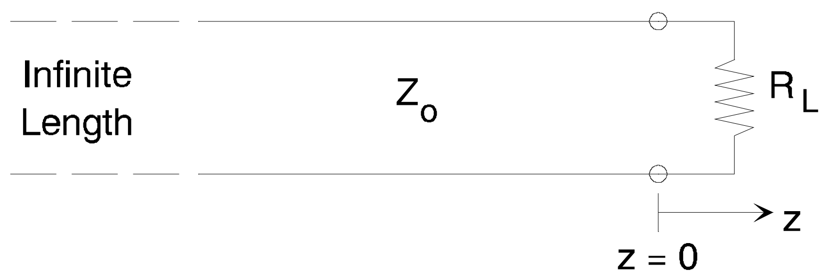


Figure 1.4: Terminated line circuit diagram

- **Note**: As a matter of convenience the reference point $z = 0$ has been shifted to the load end of the line

- Suppose that a wave traveling in the z^+ direction is incident upon the load, R_L, at $z = 0$

- Thus,

$$v(z, t) = v^+\left(t - \frac{z}{\nu_p}\right) \text{ and } i(z, t) = \frac{1}{Z_0} v^+\left(t - \frac{z}{\nu_p}\right) \tag{1.27}$$

- By applying Ohm's law at the load, we must have

$$v(0, t) = R_L i(0, t) \tag{1.28}$$

- This condition cannot, in general, be met by the incident wave alone since

$$v(z, t) = Z_0 i(z, t) \tag{1.29}$$

- Since the line was initially discharged, it is reasonable to assume that a fraction, Γ_L, of the incident wave is reflected from the load resistance, i.e.,

$$v^-(t) = \Gamma_L v^+(t) \tag{1.30}$$

- The load voltage is now

$$v(0, t) = v^+(t) + v^-(t) = (1 + \Gamma_L)v^+(t) \tag{1.31}$$

- Similarly the load current is

$$i(0, t) = \frac{v^+(t)}{Z_0} - \frac{v^-(t)}{Z_0} = \frac{(1 + \Gamma_L)v^+(t)}{Z_0} \tag{1.32}$$

- To satisfy Kirchoff's laws,

$$\frac{\text{Net load voltage}}{\text{Net load current}} = R_L = Z_0\frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{1.33}$$

or the more familiar result

$$\boxed{\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}} \tag{1.34}$$

– **Note**: Γ_L is real by assumption

– **Note**: To terminate the line without reflection use $R_L = Z_0$.

General Transmission Line Problem

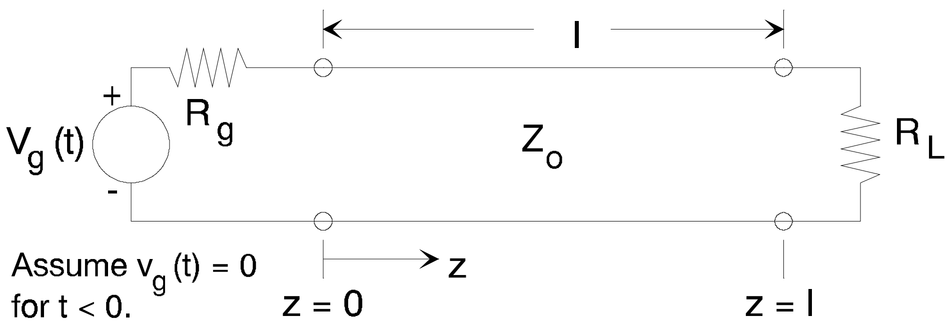


Figure 1.5: Circuit for general transmission line problem

- From earlier analysis, we know that for $0 \le t \le l/\nu_p = T_l$,

$$\left.\begin{aligned} v(z, t) &= \frac{Z_0}{Z_0 + R_g} v_g\left(t - \frac{z}{\nu_p}\right) \\ i(z, t) &= \frac{1}{Z_0 + R_g} v_g\left(t - \frac{z}{\nu_p}\right) \end{aligned}\right\} \quad 0 \le t \le T_l \tag{1.35}$$

- When $t = T_l$ the leading edge of $v_g(t)$ has traveled to the load end of the line ($z = l$)

- Assuming $R_L \ne Z_0$ a reflected wave now returns to the source during the interval $T_l \le t < 2T_l$

$$v(z, t) = \underbrace{\frac{Z_0}{Z_0 + R_g} v_g\left(t - \frac{z}{\nu_p}\right)}_{v^+(t - z/\nu_p)} + \underbrace{\frac{Z_0}{Z_0 + R_g} \Gamma_L v_g\left(t - \frac{2l}{\nu_p} + \frac{z}{\nu_p}\right)}_{v^-(t + z/\nu_p)} \tag{1.36}$$

and

$$i(z,t) = \frac{1}{Z_0+R_g}v_g\left(t-\frac{z}{\nu_p}\right) - \frac{1}{Z_0+R_g}\Gamma_L v_g\left(t-\frac{2l}{\nu_p}+\frac{z}{\nu_p}\right) \quad (1.37)$$

- When the load reflected wave $v^-(t+z/\nu_p)$ arrives at the source ($z = 0$), a portion of it will be reflected towards the load provided $R_g \neq Z_0$
- The reflection that takes place is independent of the source voltage
- The wave traveling in the positive Z direction after $2l/\nu_p = 2T_l$ seconds has elapsed, can be found by applying Ohm's law for $z = 0$ and $t = 2T_l$:

$$v_g(2T_l) - v(0, 2T_l) = R_g i(0, 2T_l) \quad (1.38)$$

- Now substitute

$$\begin{aligned} v(0, 2T_l) &= v^+(2T_l) + v^-(2T_l) \\ i(0, 2T_l) &= \frac{1}{Z_0}[v^+(2T_l) - v^-(2T_l)] \end{aligned} \quad (1.39)$$

and solve for $v^+(2T_l)$

- The results is

$$\begin{aligned} v^+(2T_l) &= \underbrace{v_g(2T_l)\frac{Z_0}{Z_0+R_g}}_{\text{incident part of } v_g(t)} + \underbrace{v^-(2T_l)\frac{R_g - Z_0}{R_g+Z_0}}_{\text{reflected portion of } v^-(2T_l)} \\ &= v_g(2T_l)\frac{Z_0}{Z_0+R_g} + v_g(0)\frac{Z_0}{Z_0+R_g}\Gamma_L\Gamma_g \end{aligned} \quad (1.40)$$

where Γ_L is the source reflection coefficient defined as

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} \tag{1.41}$$

Note: Γ_g is real by assumption in this case

- The wave incident on the load during the interval $2T_l \le t < 3T_l$, thus consists of the original source signal plus a reflected component due to mismatches at both the load and source ends of the line

$$\begin{aligned} v(z, t) = \frac{Z_0}{Z_0 + R_g}\Big[v_g\Big(t - \frac{z}{\nu_p}\Big) + \Gamma_L v_g\Big(t + \frac{z - 2l}{\nu_p}\Big) \\ + \Gamma_g \Gamma_L v_g\Big(t - \frac{z + 2l}{\nu_p}\Big)\Big], 2T_l \le t < 3T_l \end{aligned} \tag{1.42}$$

and

$$\begin{aligned} i(z, t) = \frac{1}{Z_0 + R_g}\Big[v_g\Big(t - \frac{z}{\nu_p}\Big) - \Gamma_L v_g\Big(t + \frac{z - 2l}{\nu_p}\Big) \\ + \Gamma_g \Gamma_L v_g\Big(t - \frac{z + 2l}{\nu_p}\Big)\Big], 2T_l \le t < 3T_l \end{aligned} \tag{1.43}$$

- The process of reflections occurring at both the source and load ends continues in such a way that in general over the n^{th} time interval, $(n - 1)T_l \le t \le nT_l$, the $v(z, t)$ and $i(z, t)$ solutions each require n terms involving $v_g(t)$

Example: Consider the following circuit.

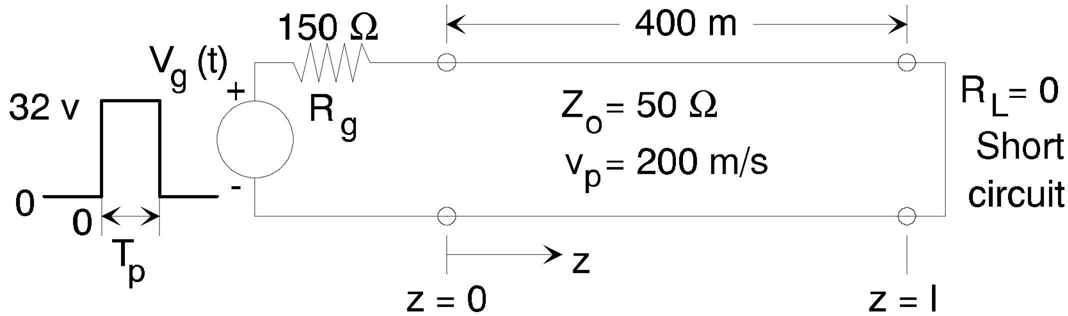


Figure 1.6: Transmission line circuit

Here we have

$$\frac{Z_0}{Z_0 + R_g} = \frac{1}{4}, \Gamma_L = -1, \Gamma_g = \frac{1}{2}, T_l = 2\mu\text{s} \qquad (1.44)$$

a) Find $V(0, t)$ and $I(0, t)$ for $T_p = 1\mu\text{s}$

b) Find $V(0, t)$ and $I(0, t)$ for $T_p = 6\mu\text{s}$

- For both parts (a) and (b) the basic circuit operation is as follows:
  i) An 8v pulse will propagate toward the load, reaching the load in 2 μs.
  ii) A -8v pulse will be reflected from the load, requiring 2 μs to reach the source.
  iii) A -4v pulse will be reflected at the source. It will take 2 μs to reach the load.
  iv) The process continues.
- The +*z* and -*z* direction propagating pulses can be displayed on a distance-time plot or bounce diagram (see Figure 1.7)

– In the bounce diagram the boundaries at $z = 0$ and $z = 400$ m are represented as surfaces with reflection coefficient of 1/2 and -1 respectively.

– To obtain the voltage waveform at say $z = 380$ m as a function of time, you sum the contributions from the various $+z$ and $-z$ traveling waves

– For pulses that are short in comparison with the one-way delay time of the line, only at most two wave terms need to be included at a time.

– For long pulses (in the limit say a step function) all wave terms need to be included

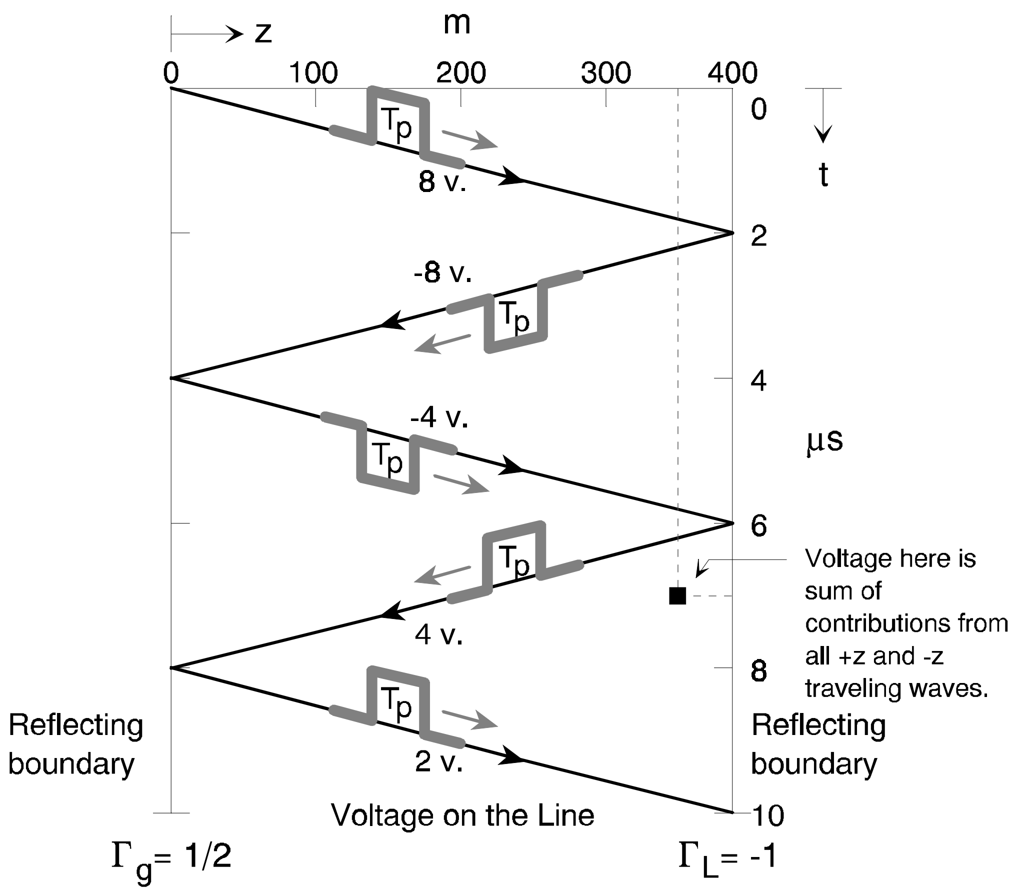


Figure 1.7: Bounce diagram showing pulse propagation

a) $T_p = 1\mu\text{s}$: at $t = 4\mu\text{s}$ we have

$$v(0, 4\mu\text{s}) = -4 + (-8) = -12\text{v}.$$

$$i(0, 4\mu\text{s}) = \frac{1}{50}[-4 - (-8)] = 0.08\text{A}.$$

b) $T_p = 6\mu\text{s}$: at $t = 4\mu\text{s}$ we have

$$v(0, 4\mu\text{s}) = (8 + (-4)) + (-8) = -4\text{v}.$$

$$i(0, 4\mu\text{s}) = \frac{1}{50}[(8 + (-4)) - (-8)] = 0.24\text{A}.$$

- We can simulate this result using the Agilent *Advanced Design System* (ADS) software
- In this example we will use ideal transmission line elements

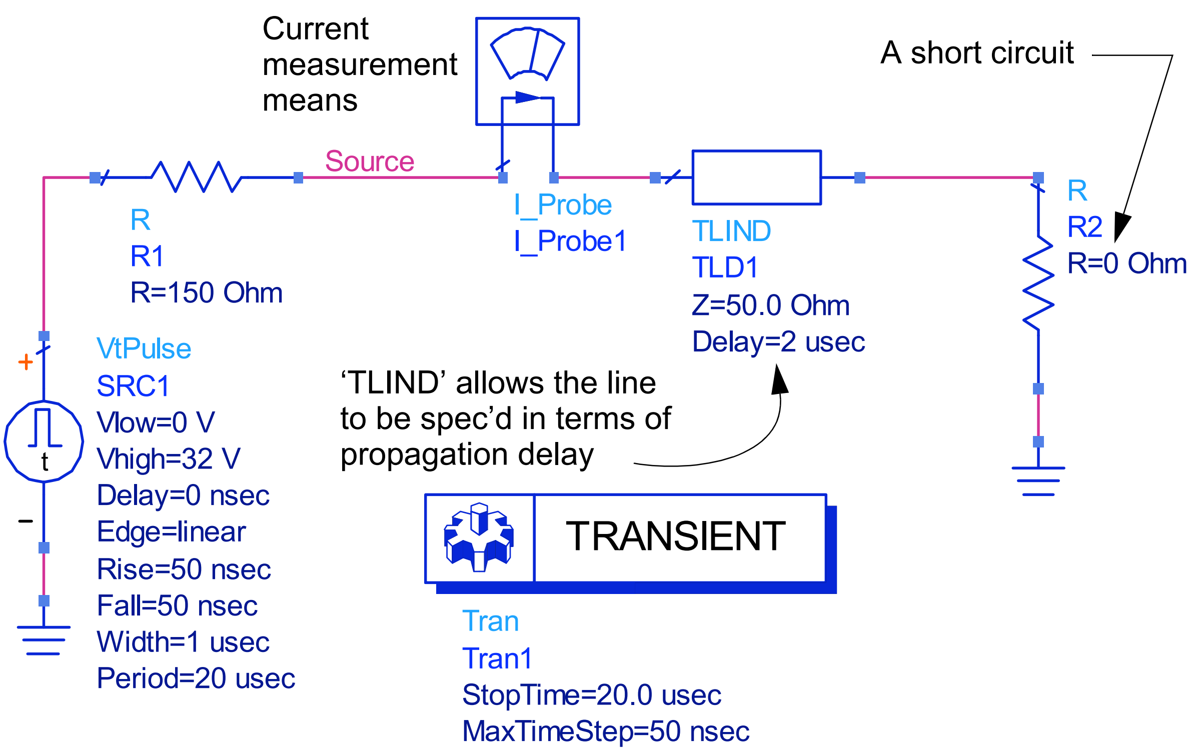
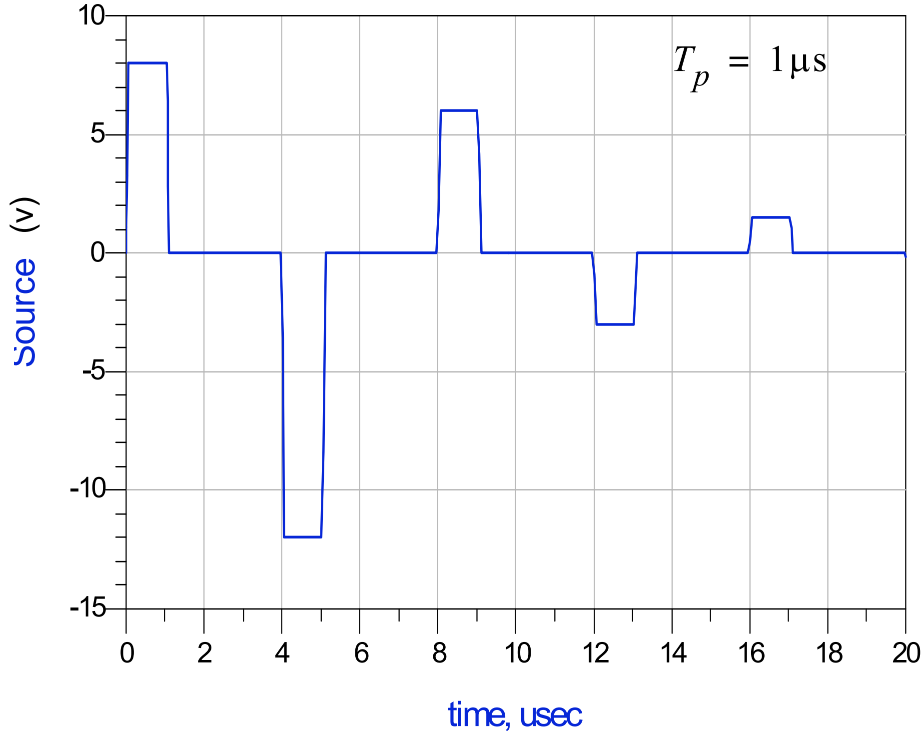
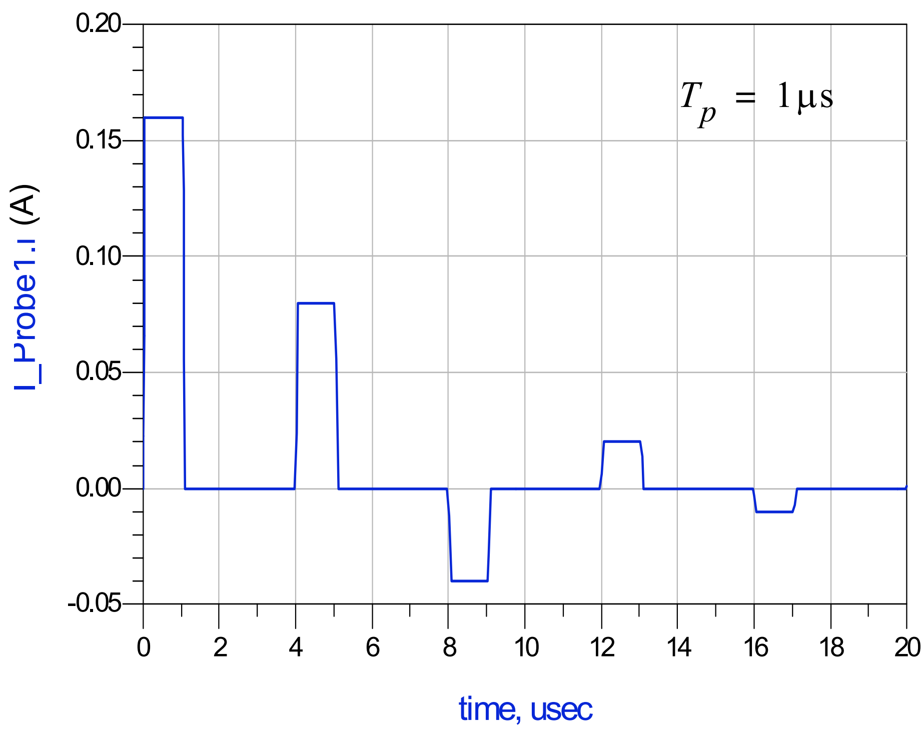


Figure 1.8: Circuit schematic (Example1.dsn)

- The source end voltage, $v(0, t)$

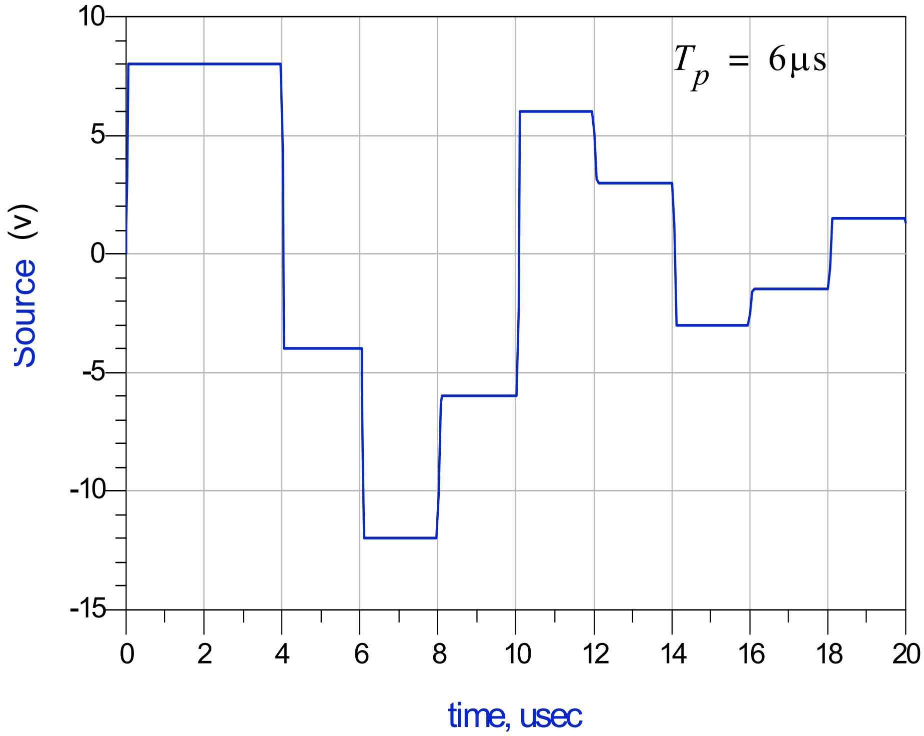


- The source end current entering the line, $i(0, t)$

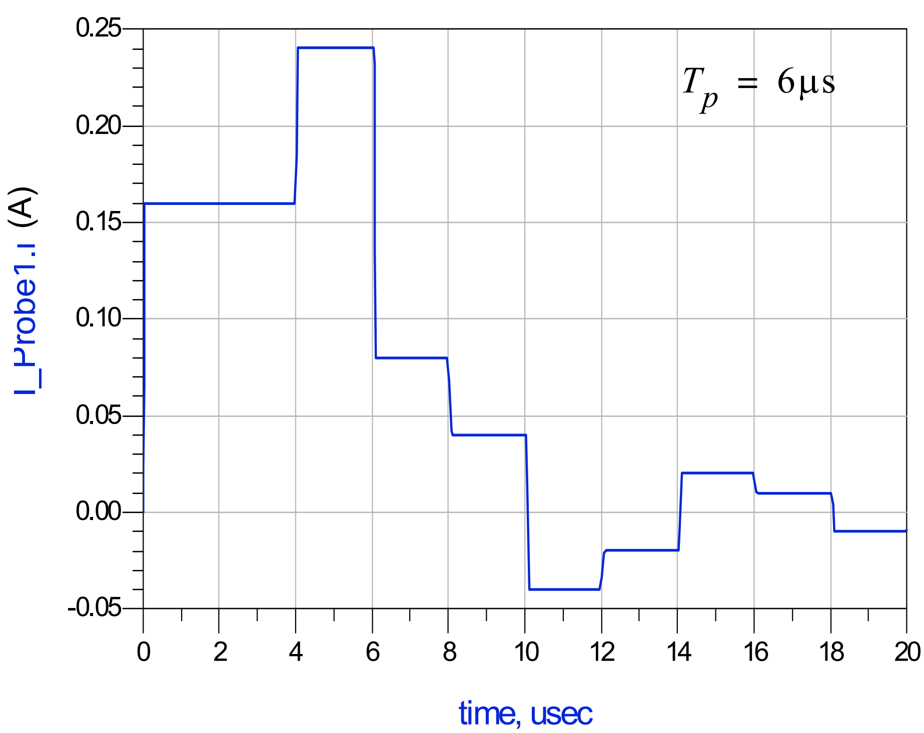


- Modify the schematic by increasing the pulse width to 6 μs

- The source end voltage, $v(0, t)$



- The source end current entering the line, $i(0, t)$



Example 2.21: LTspice Extension to ADS Example

- ADS (from Keysight) is a far more complex tool than LTspice
- This example repeats and then extends some of the content of the ADS example
- The schematic is shown below for pulse width of 1μs

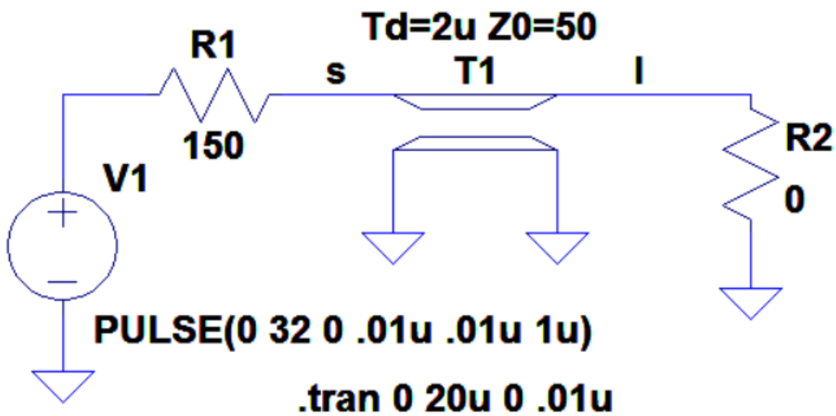


Figure 2.93: LTspice schematic of a simple transient response example.

- We can plot the total voltage, incident, and reflected voltages at $z = 0$
- By breaking out the incident and reflected voltage waves the math becomes clear

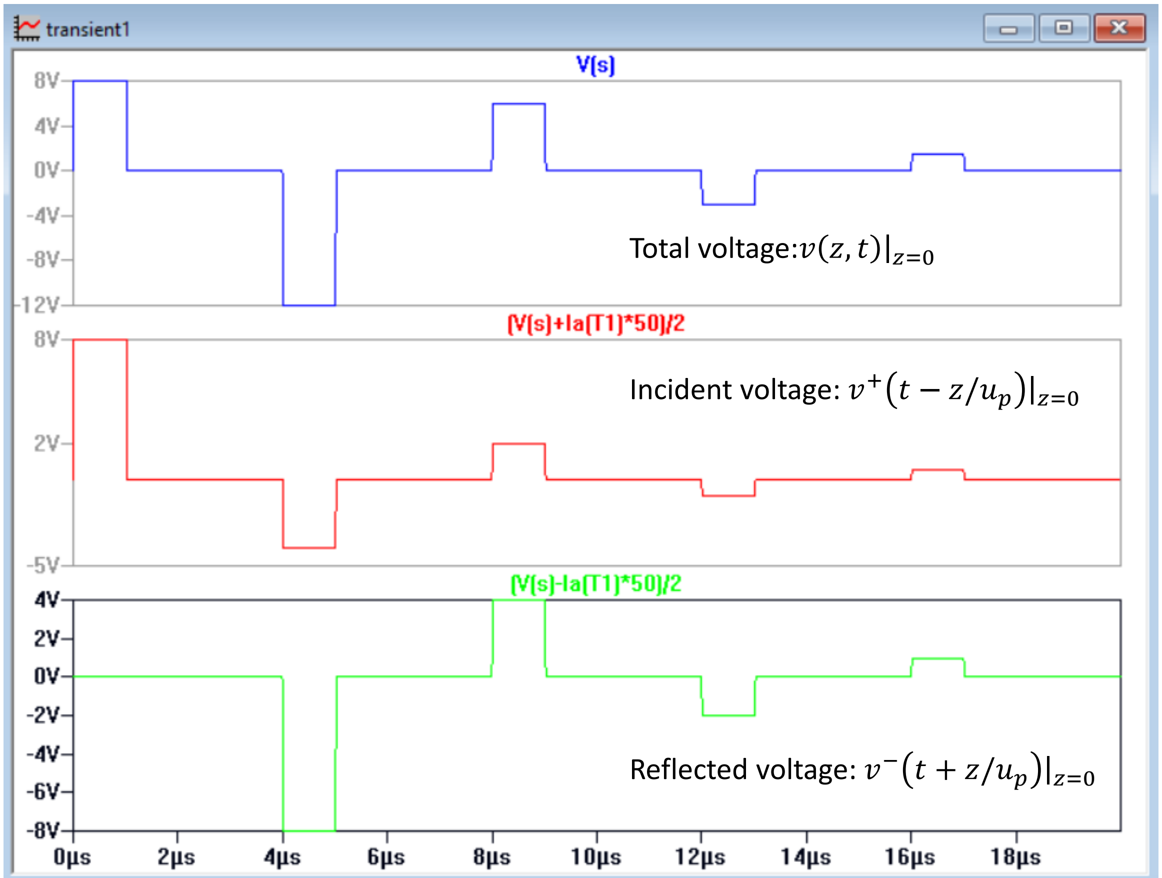


Figure 2.94: Voltage at $z = 0$ for a $1\mu s$ pulse launched on the line.

Transient Analysis using Laplace Transforms

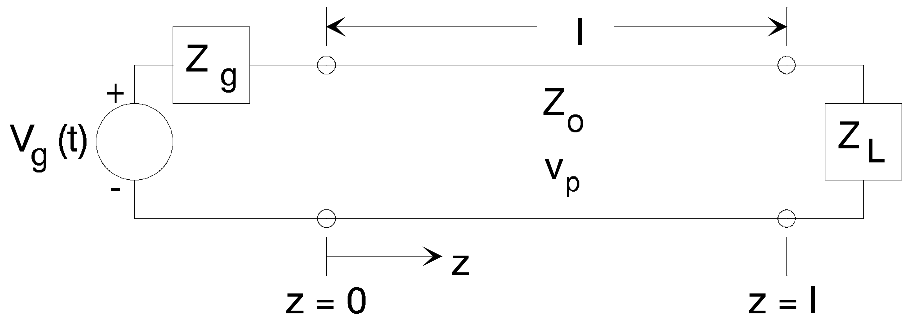


Figure 1.9: Lossless line with arbitrary terminations

- In the time domain we know that for a lossless line

$$v(z, t) = v^{+}\left(t - \frac{z}{\nu_p}\right) + v^{-}\left(t + \frac{z}{\nu_p}\right) \tag{1.45}$$

  and

$$i(z, t) = \frac{1}{Z_0}\left[v^{+}\left(t - \frac{z}{\nu_p}\right) - v^{-}\left(t + \frac{z}{\nu_p}\right)\right] \tag{1.46}$$

  where $v^{+}(t - z/\nu_p)$ and $v^{-}(t + z/\nu_p)$ are determined by the boundary conditions imposed by the source and load

- Laplace transform each side of the above equations with respect to the time variable, using the time shift theorem which is given by

$$\mathcal{L}\{f(t - c)\} = F(s)e^{-sc} \tag{1.47}$$

  where $F(s)$ is the laplace transform of $f(t)$

- The result is

$$v(z, s) = v^{+}(s)e^{-sz/\nu_p} + v^{-}(s)e^{sz/\nu_p} \tag{1.48}$$

  and

$$i(z, s) = \frac{1}{Z_0}[v^+(s)e^{-sz/v_p} - v^-(s)e^{sz/v_p}] \tag{1.49}$$

where $v^+(s) = \mathcal{L}\{v^+(t)\}$ and $v^-(s) = \mathcal{L}\{v^-(t)\}$

Case 1: Matched Source

- For the special case of $Z_g(s) = Z_0$, the source is matched to the transmission line which eliminates multiple reflections
- Thus, we can write

$$V^+(s) = V_g(s)\frac{Z_0}{Z_0 + Z_0} = \frac{1}{2}V_g(s) \tag{1.50}$$

at $z = l$, the incident wave is reflected with the reflection coefficient

$$\Gamma_L(s) = \frac{Z_L(s) - Z_0}{Z_L(s) + Z_0} \tag{1.51}$$

so,

$$V(l, s) = V^+(s)e^{-sl/v_p}[1 + \Gamma_L(s)] \tag{1.52}$$

which implies that

$$V^-(s) = \Gamma_L(s)e^{-s2l/v_p}V^+(s) \tag{1.53}$$

- Finally, for $0 \le z \le l$ we can write

$$\begin{aligned} V(z, s) &= \frac{1}{2}V_g(s)[e^{-sz/v_p} + \Gamma_L(s)e^{-s(2l - z)/v_p}] \\ &= \frac{1}{2}V_g(s)e^{-sz/v_p} + \frac{1}{2}\Gamma_L(s)e^{-s2l/v_p}V_g(s)e^{sz/v_p} \end{aligned} \tag{1.54}$$

- To obtain $V(z, t)$ inverse transform:

$$v(z, t) = \frac{1}{2} v_g\left(t - \frac{z}{\nu_p}\right) + \mathcal{L}^{-1}[\Gamma_L(s) V_g(s)]\Big|_{t \to t + (z - 2l)/\nu_p} \tag{1.55}$$

Example:

- Let $v_g(t) = v_0 u(t)$ and Z_L *be* a parallel RC connection

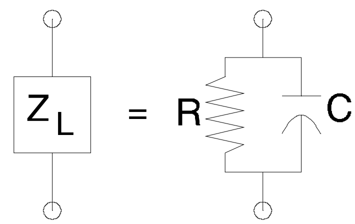


Figure 1.10: Parallel *RC* circuit

Find: $v(z, t)$

- To begin with in the s-domain we can write

$$Z_L(s) = \frac{R\dfrac{1}{Cs}}{R + \dfrac{1}{Cs}} = \frac{R}{1 + RCs} \tag{1.56}$$

and

$$\begin{aligned} \Gamma_L(s) &= \frac{Z_L(s) - Z_0}{Z_L(s) + Z_0} = \frac{\dfrac{R}{1 + RCs} - Z_0}{\dfrac{R}{1 + RCs} + Z_0} \\ &= \frac{\dfrac{R - Z_0}{RCZ_0} - s}{\dfrac{R - Z_0}{RCZ_0} + s} = \frac{b - s}{a + s}, \; a = \frac{R + Z_0}{RCZ_0}, \; b = \frac{R - Z_0}{RCZ_0} \end{aligned} \tag{1.57}$$

- Now since $V_g(s) = v_0 \mathcal{L}\{u(t)\} = v_0/s$

$$V(z, s) = \frac{v_0}{2s}\left[e^{-sz/\nu_p} + \frac{b-s}{a+s}e^{-s(2l-z)/\nu_p}\right] \tag{1.58}$$

- To inverse transform first apply partial fractions to

$$\frac{b-s}{s(a+s)} = \frac{K_1}{s} + \frac{K_2}{s+a} \tag{1.59}$$

Clearly,

$$K_1 = \frac{b}{a} = \frac{R-Z_0}{R+Z_0} \qquad K_2 = \frac{-(a+b)}{a} = \frac{-2R}{R+Z_0} \tag{1.60}$$

so

$$\begin{aligned} V(z, s) = \frac{v_0}{2}\Bigg[&\frac{1}{s}e^{-sz/\nu_p} \\ &+ \left\{\frac{R-Z_0}{R+Z_0}\cdot\frac{1}{s} - \frac{2R}{R+Z_0}\cdot\frac{1}{s+a}\right\}e^{-\frac{s(2l-z)}{\nu_p}}\Bigg] \end{aligned} \tag{1.61}$$

and

$$\begin{aligned} v(z, t) = L^{-1}\{V(z, s)\} = \frac{v_0}{2}\Bigg[&u\left(t - \frac{z}{\nu_p}\right) + \Bigg\{\frac{R-Z_0}{R+Z_0} \\ &- \frac{2R}{R+Z_0}e^{-\left(t - \frac{2l-z}{\nu_p}\right)\frac{R+Z_0}{RCZ_0}}\Bigg\}u\left(t - \frac{2l-z}{\nu_p}\right)\Bigg] \end{aligned} \tag{1.62}$$

- As a special case consider $z = l$

$$v(z,t) = \frac{v_0}{2}\left[1 + \frac{R - Z_0}{R + Z_0} - \frac{2R}{R + Z_0} e^{-\left(t - \frac{l}{\nu_p}\right)\frac{R + Z_0}{RCZ_0}}\right] u\left(t - \frac{l}{\nu_p}\right)$$

$$= v_0 \frac{R}{R + Z_0}\left[1 - e^{-\left(t - \frac{l}{\nu_p}\right)\frac{R + Z_0}{RCZ_0}}\right] u\left(t - \frac{l}{\nu_p}\right) \tag{1.63}$$

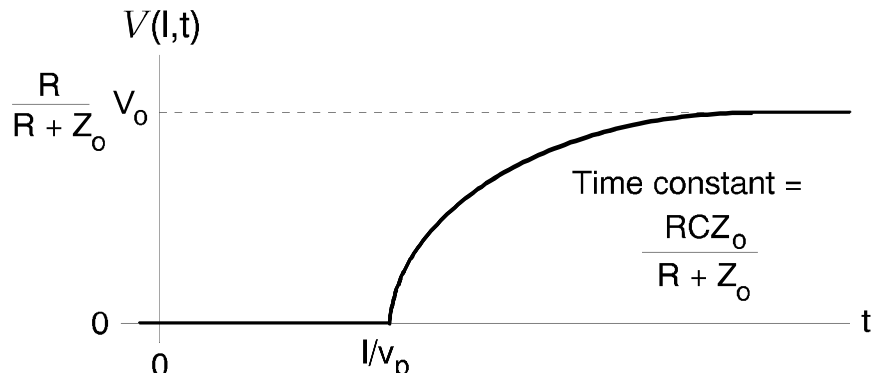


Figure 1.11: Sketch of the general $v(l, t)$ waveform

Example: The results of an ADS simulation when using $v_0 = 2v$, $Z_0 = R = 50$ ohms, $C = 50$ pf, and $T_l = 5$ ns is shown below

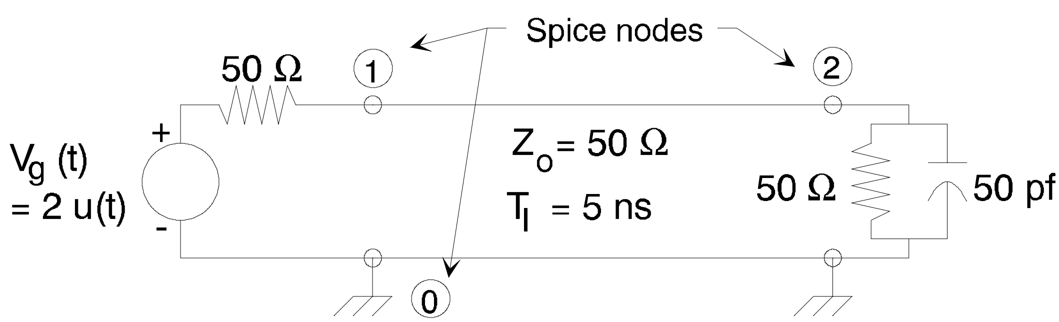
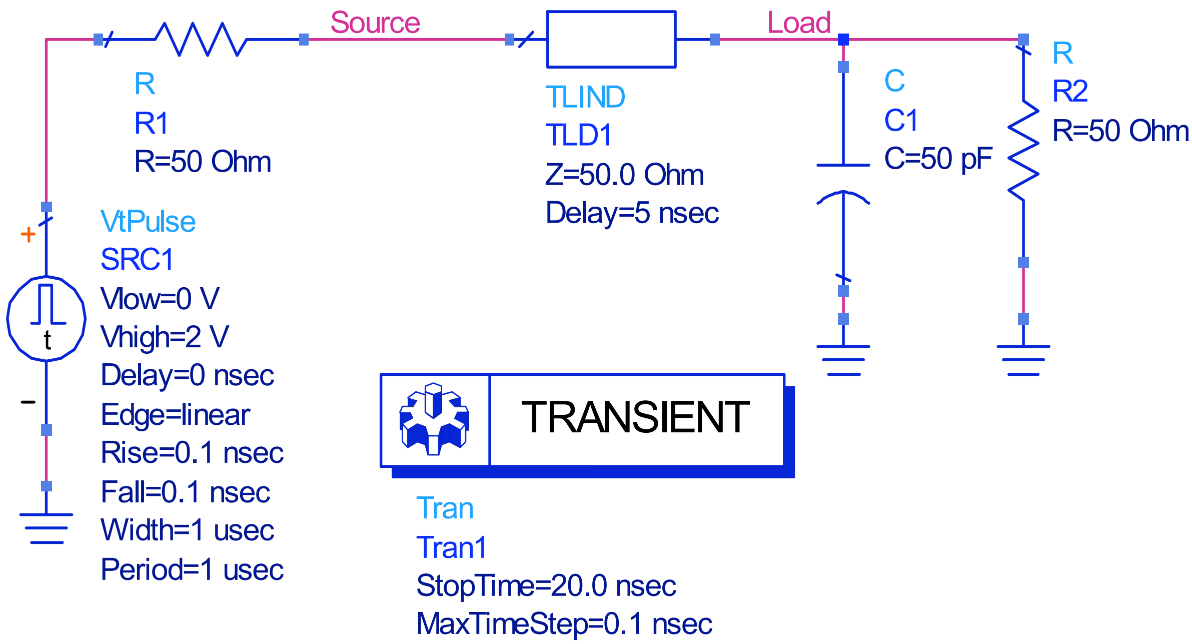
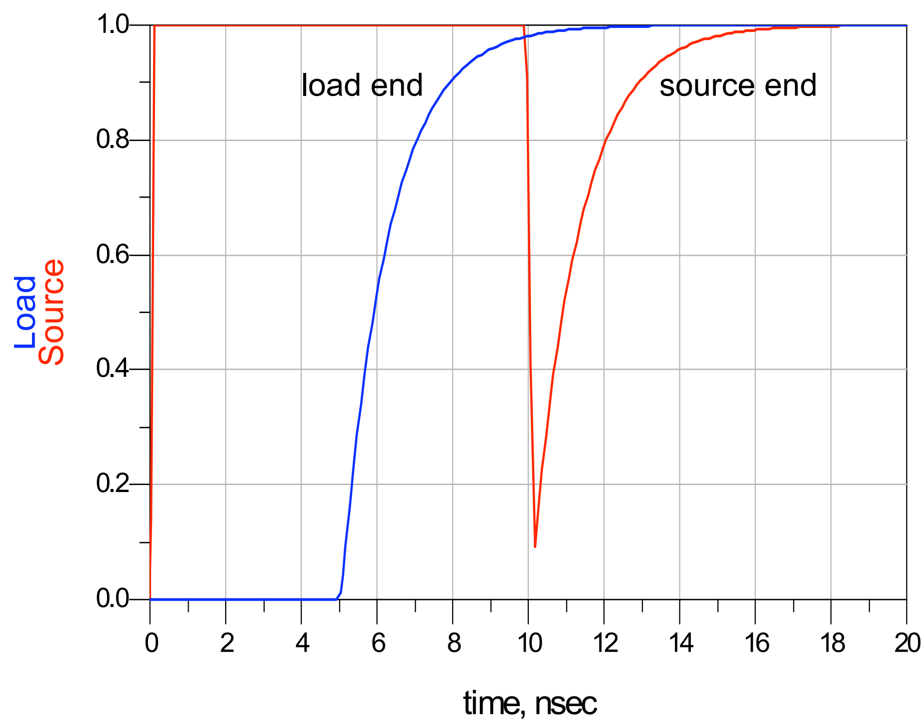


Figure 1.12: ADS Circuit diagram with nodes numbered

- The ADS schematic is shown below



- Next we plot the source and load end waveforms



Case 2: Mismatched Source Impedance

- For the general case where $Z_g(s) \neq Z_0$ to reflections occur at the source end of the line as well as at the load end
- The expression for $V(z, s)$ now will consist of an infinite number of terms as shown below:

$$\begin{aligned} V(z, s) = V_g(s)\frac{Z_0}{Z_0 + Z_g(s)}[&e^{-sz/v_p} + \Gamma_L(s)e^{-s(2l - z)/v_p} \\ &+ \Gamma_L(s)\Gamma_g(s)e^{-s(2l + z)/v_p} \\ &+ \Gamma_L(s)\Gamma_g(s)\Gamma_L(s)e^{-s(4l - z)/v_p} \\ &+ \Gamma_L(s)\Gamma_g(s)\Gamma_L(s)\Gamma_g(s)e^{-s(4l + z)/v_p} + \cdots] \end{aligned} \tag{1.64}$$

where

$$\Gamma_L(s) = \frac{Z_L(s) - Z_0}{Z_L(s) + Z_0} \text{ and } \Gamma_g(s) = \frac{Z_g(s) - Z_0}{Z_g(s) + Z_0} \tag{1.65}$$

- In terms of $+z$ and $-z$ propagating waves we can write

$$\begin{aligned} V(z, s) = \frac{V_g(s)Z_0}{Z_0 + Z_g(s)}\Bigg[&e^{-sz/vp}\sum_{n=0}^{\infty}\Gamma_L^n(s)\Gamma_g^n(s)e^{-s(2n)l/v_p} \\ &+ e^{sz/v_p}\sum_{n=0}^{\infty}\Gamma_L^{n+1}(s)\Gamma_g^n(s)e^{-s(2n+2)l/v_p}\Bigg] \end{aligned} \tag{1.66}$$

Example: Consider a circuit with $v_0 = 2\text{v}$, $Z_0 = 50$ ohms, $T_l = 5$ ns, $Z_g = R_g = 100$ ohms, and Z_L a parallel RC circuit with $R = 100$ ohms and $C = 20$ pf, as shown below in Figure 1.13.

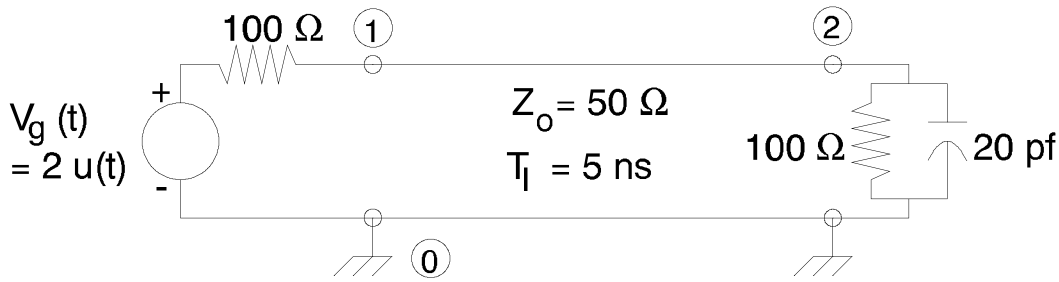


Figure 1.13: Circuit diagram with Spice nodes indicated

- In the s-domain the solution is of the form

$$V(z, s) = \frac{2}{3}\left[e^{-sz/v_p}\sum_{n=0}^{\infty}\frac{(b-s)^n}{s(s+a)^n}\left(\frac{1}{3}\right)^n e^{-s(2n)l/v_p}\right.$$
$$\left. + e^{sz/v_p}\sum_{n=0}^{\infty}\frac{(b-s)^{n+1}}{s(s+a)^{n+1}}\left(\frac{1}{3}\right)^n e^{-s(2n+2)l/v_p}\right] \quad (1.67)$$

- To inverse transform $V(z, s)$ note that each series term consists of the product of a constant, a ratio of polynomials in s, and a time shift exponential (i.e. $e^{-s\tau}$)

- In the time-domain each series term to within a constant is of the form

$$\mathcal{L}^{-1}\left\{\frac{(b-s)^n}{s(s+a)^n}\right\}\Bigg|_{t \to t-\tau_n}, n = 0, 1, 2, \ldots \quad (1.68)$$

- A partial fraction expansion of the ratio of polynomials in (1.68) is

$$\frac{(b-s)^n}{s(s+a)^n} = \frac{K_1}{s} + \frac{K_{12}}{s+a} + \frac{K_{22}}{(s+a)^2} + \cdots + \frac{K_{2n}}{(s+a)^n} \tag{1.69}$$

where

$$K_1 = \frac{b^n}{a^n} \tag{1.70}$$

and

$$K_{2k} = \frac{1}{(n-k)!}\frac{d^{(n-k)}}{ds^{(n-k)}}\left[\frac{(b-s)^n}{s}\right]\Bigg|_{s=-a}, k = 1, 2, \ldots, n \tag{1.71}$$

- To obtain a partial solution for comparison with a Spice simulation we will solve (1.69) for $n = 0$, 1, and 2.
  - Case $n = 0$:

$$\frac{1}{s} \Leftrightarrow u(t) \tag{1.72}$$

  - <u>*Case n* = 1</u>

$$\frac{b-s}{s(s+a)} = \frac{b/a}{s} - \frac{(a+b)/a}{s+a} \Leftrightarrow \left[\frac{b}{a} - \frac{a+b}{a}e^{-at}\right]u(t) \tag{1.73}$$

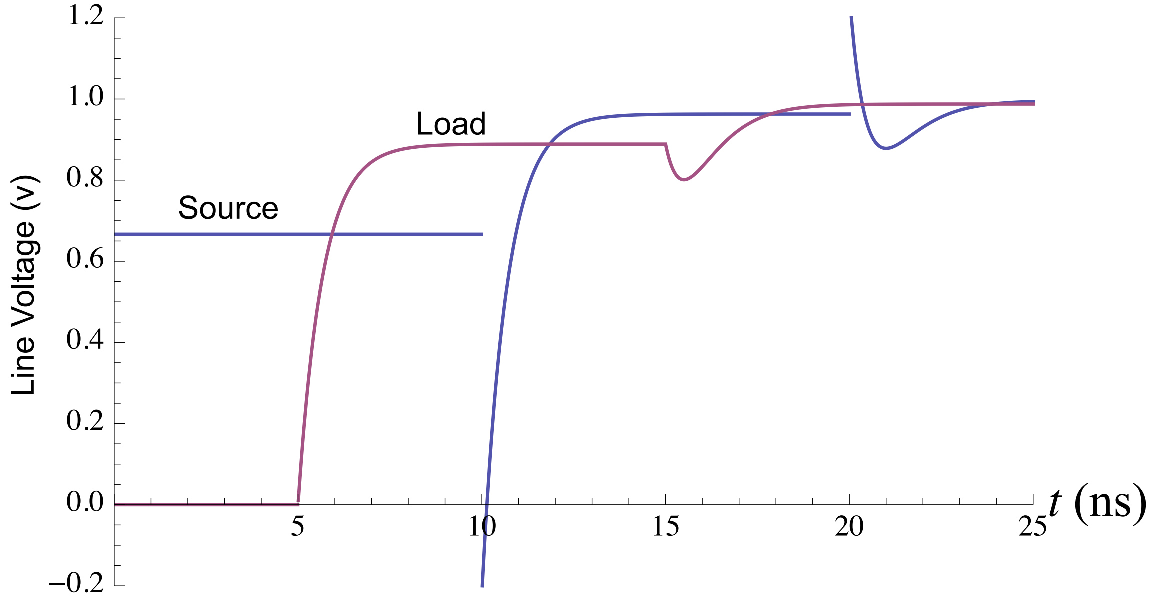
– $\underline{n = 2}$

$$\frac{(b-s)^2}{s(s+a)^2} = \frac{b^2/a^2}{s} + \frac{(1-b^2/a^2)}{s+a} - \frac{(b+a)^2/a}{(s+a)^2}$$
$$\Leftrightarrow \left[\frac{b^2}{a^2} + \left(1 - \frac{b^2}{a^2}\right)e^{-at} - \frac{(b+a)^2}{a}te^{-at}\right]u(t) \tag{1.74}$$

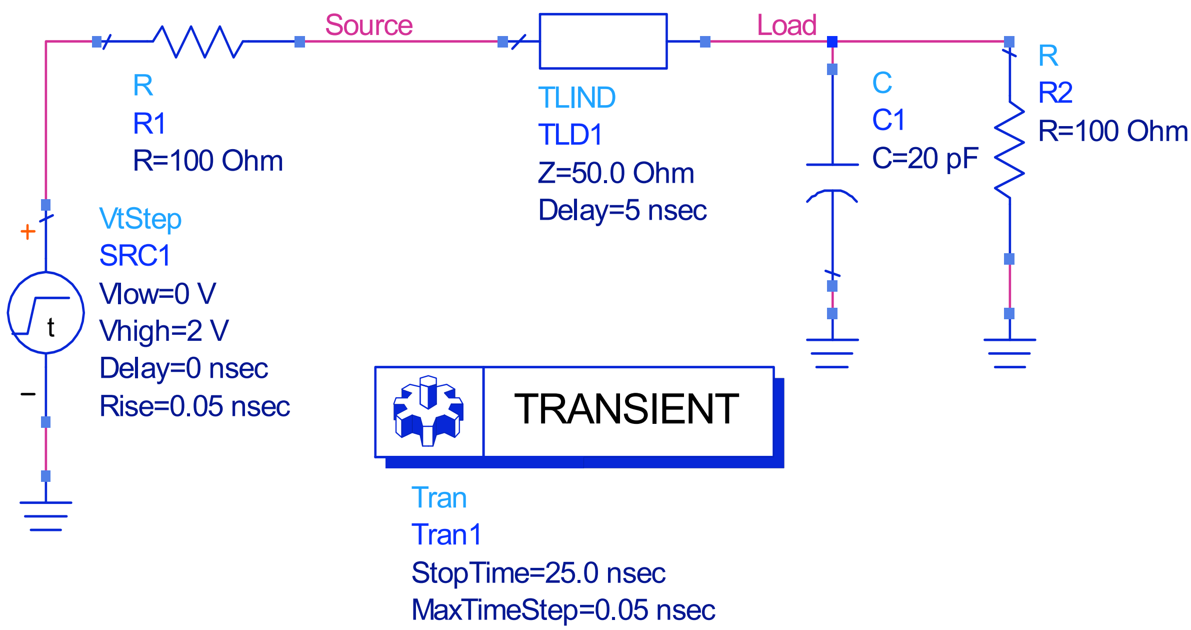
- Using Mathematica the analytical solution valid for t up to 25 ns was obtained

```
a = (RL + Z0)/(RL C Z0) /. {RL → 100, Z0 → 50, C → 20 × 10^-12};

b = (RL - Z0)/(RL C Z0) /. {RL → 100, Z0 → 50, C → 20 × 10^-12};

v0[t_] := 2/3 UnitStep[t];

v1[t_] := 2/3 (b/a - (a + b)/a ⅇ^(-a t/10^9)) UnitStep[t]

v2[t_] := 2/3 (b^2/a^2 + (1 - b^2/a^2) ⅇ^(-a t/10^9) - (b + a)^2/a t/10^9 ⅇ^(-a t/10^9))
   UnitStep[t];

Plot[{v0[t] + v1[t - 10] + 1/3 v1[t - 10] + 1/3 v2[t - 20] +
   1/9 v2[t - 20], v0[t - 5] + 1/3 v1[t - 15] + v1[t - 5] +
   1/3 v2[t - 15]}, {t, 0, 25},
  PlotRange → {{0, 25}, {-.2 , 1.2}},
  PlotStyle → AbsoluteThickness[1]]
```

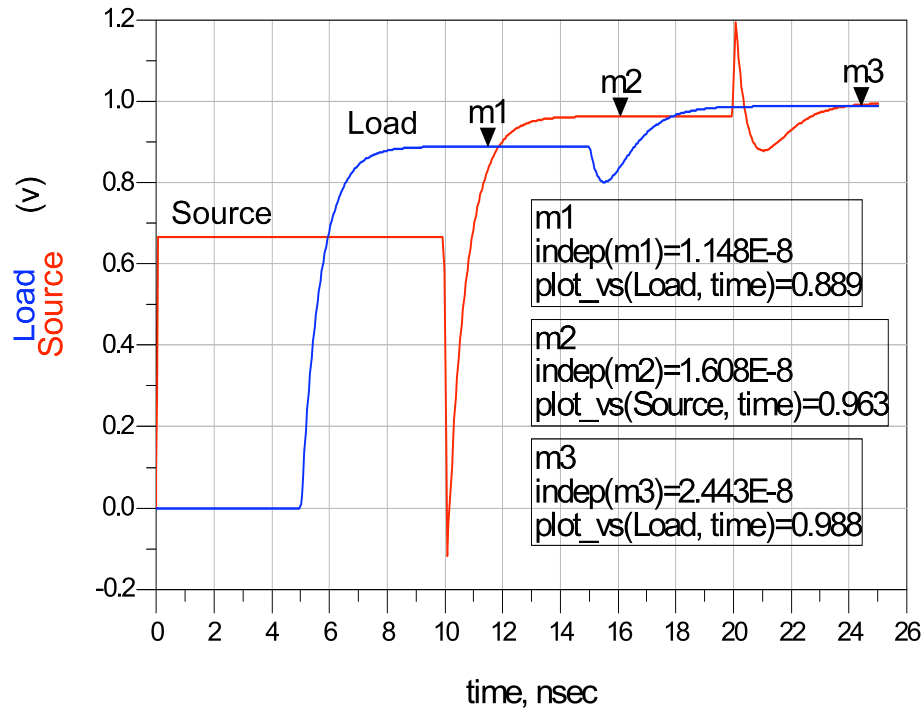
- This required the use of $+z$ and $-z$ wave solutions from the series for $n = 0$ and 1
- The theoretical voltage waveforms at $z = 0$ and $z = 1$ are shown below



- A circuit simulation using ADS was also run, the results compare favorably as expected



- Plots of $v(0, t))$ and $v(l, t)$



2.12.3 The Time-Domain Reflectometer

- A *time-domain reflectometer* (TDR) is an instrument that sends a step function down a transmission line
- The TDR is known for having a very **fast edge** or step function rise time; e.g., 10's of ps and much lower
- A fundamental use of the TDR is in looking for cable faults and PCB track discontinuities
- The TDR does all of its *detective work* from the source end of the line

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Example 2.22: **Generate Pulse From Step**

- In this example we consider a microstrip design for converting the step function output from a TDR into a rectangular pulse
- Start with a LTspice simulation to see the basis for the design

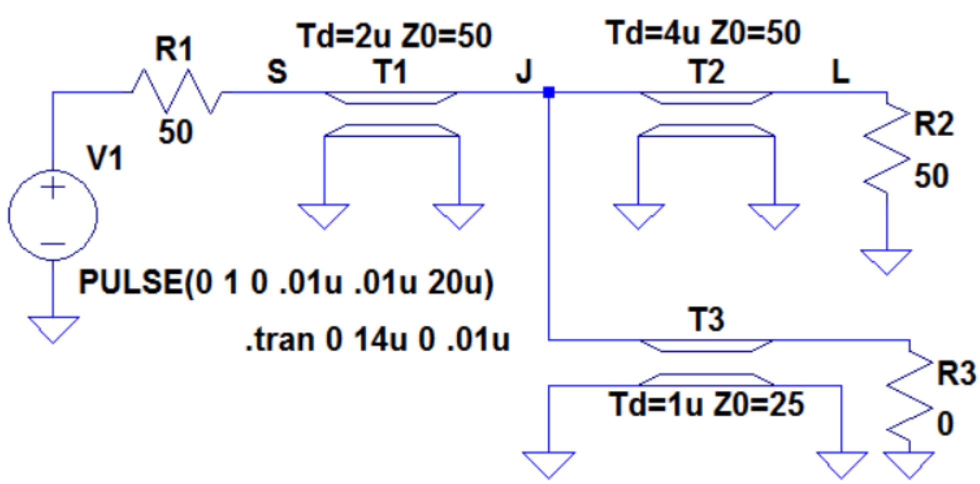

Figure 2.95: LTspice design of the step to pulse converter circuit.

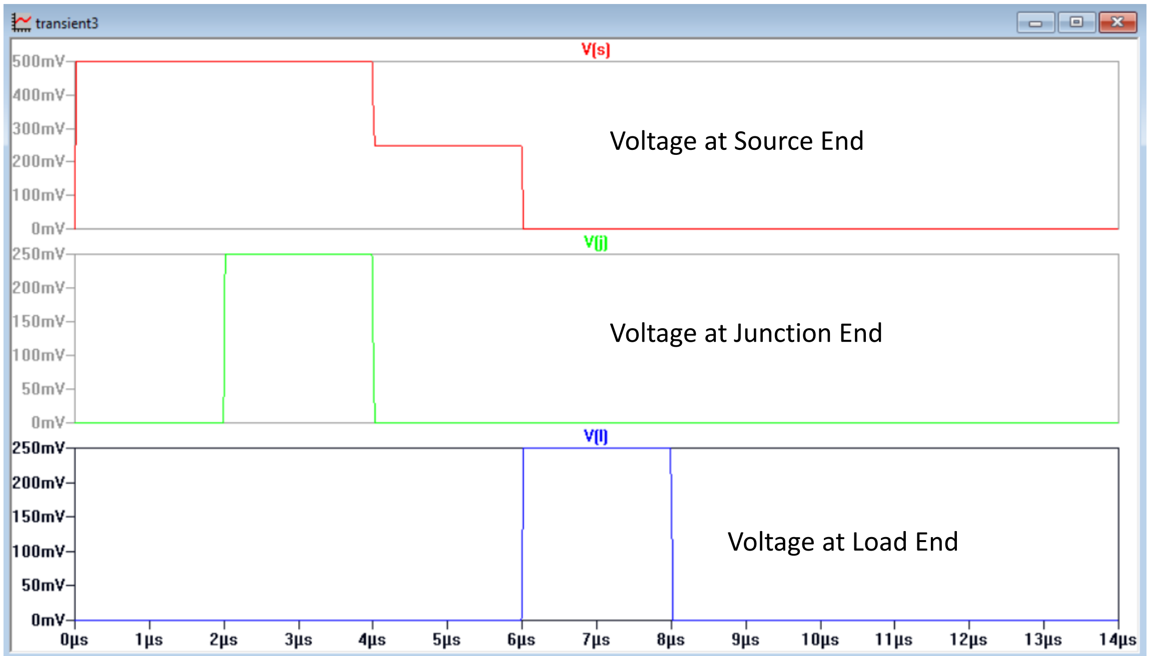

Figure 2.96: Pulse converter waveforms.

- The microstrip design places two 50 Ω stubs in parallel to form a more balanced circuit layout, that is all transmission widths are equal

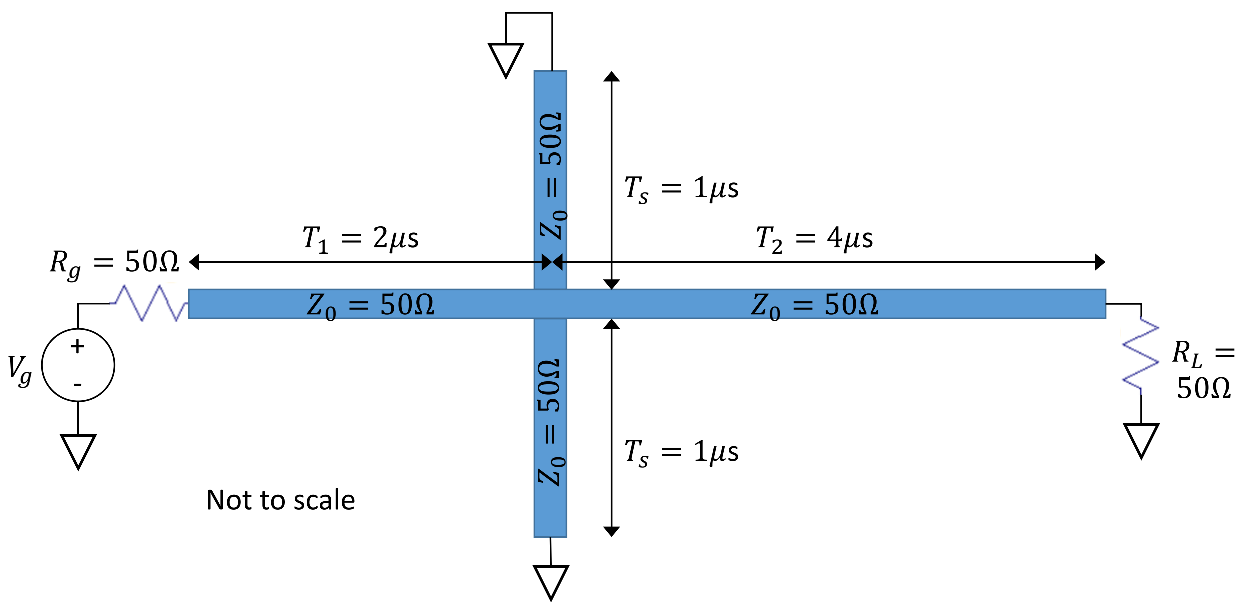

Figure 2.97: Microstrip circuit design concept.