LECTURE 03 Coordinate Systems (좌표계)

1. XY-Coordinate System
2. Polar Coordinate System
3. Rectangular Coordinate System
4. Cylindrical Coordinate System
5. Spherical Coordinate System
6. Coordinate Transform
7. Infinitesimal Surface and Volume
8. Coding Example

1. XY-Coordinate System

Coordinate: $P(x, y)$

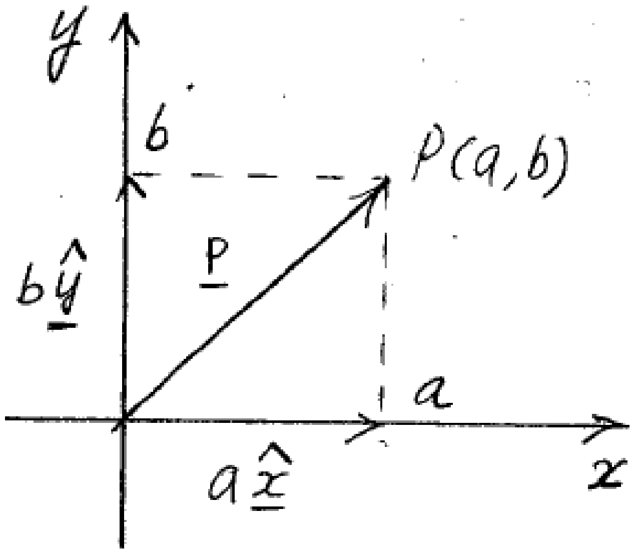
Base vectors: $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$

Differentials:

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

$$d\mathbf{L} = (dx)\hat{\mathbf{x}} + (dy)\hat{\mathbf{y}}$$

$$dS = dxdy$$

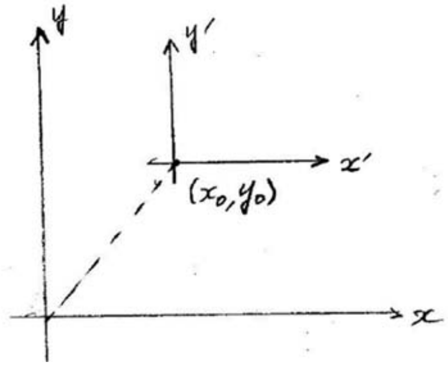


Distance:

$$R = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Vector field: $F(x, y) = F_x(x, y)\hat{\mathbf{x}} + F_y(x, y)\hat{\mathbf{y}}$

Translation: $x = x_0 + x'$, $y = y_0 + y'$



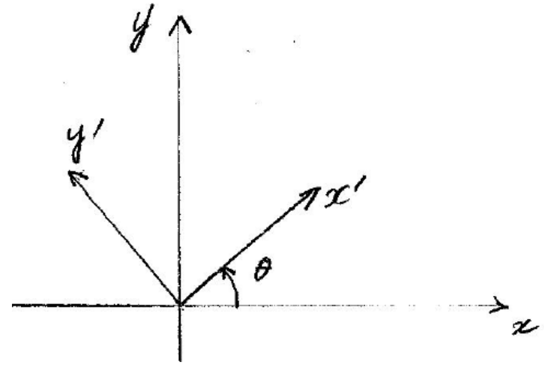
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Rotation:

$$x = x'\cos\theta + y'\cos(\theta + \pi/2) = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\sin(\theta + \pi/2) = x'\sin\theta + y'\cos\theta$$

$$\boxed{\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}$$

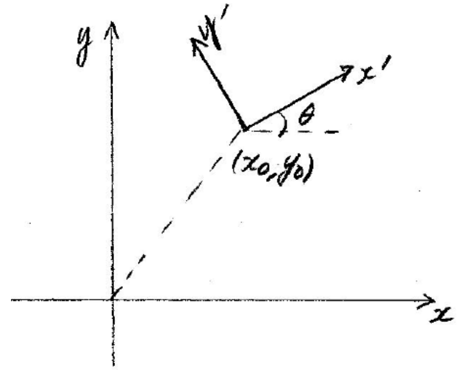


Translate-then-rotate:

$$x = x_0 + x'\cos\theta + y'\cos(\theta + \pi/2) = x'\cos\theta - y'\sin\theta$$

$$y = y_0 + x'\sin\theta + y'\sin(\theta + \pi/2) = x'\sin\theta + y'\cos\theta$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$



- **Arc length**

$$y = f(x)$$

$$L = \int dL = \int_{x=a}^{x=b} dL = \int_{x=a}^{x=b} \sqrt{(dx)^2 + (dy)^2} = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$x = f(t),\ y = g(t)$$

$$L = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example: Length of a parabolic curve

Find the arc length of a curve $y = 3x^2$ for x from 0 to 2.

(Solution)

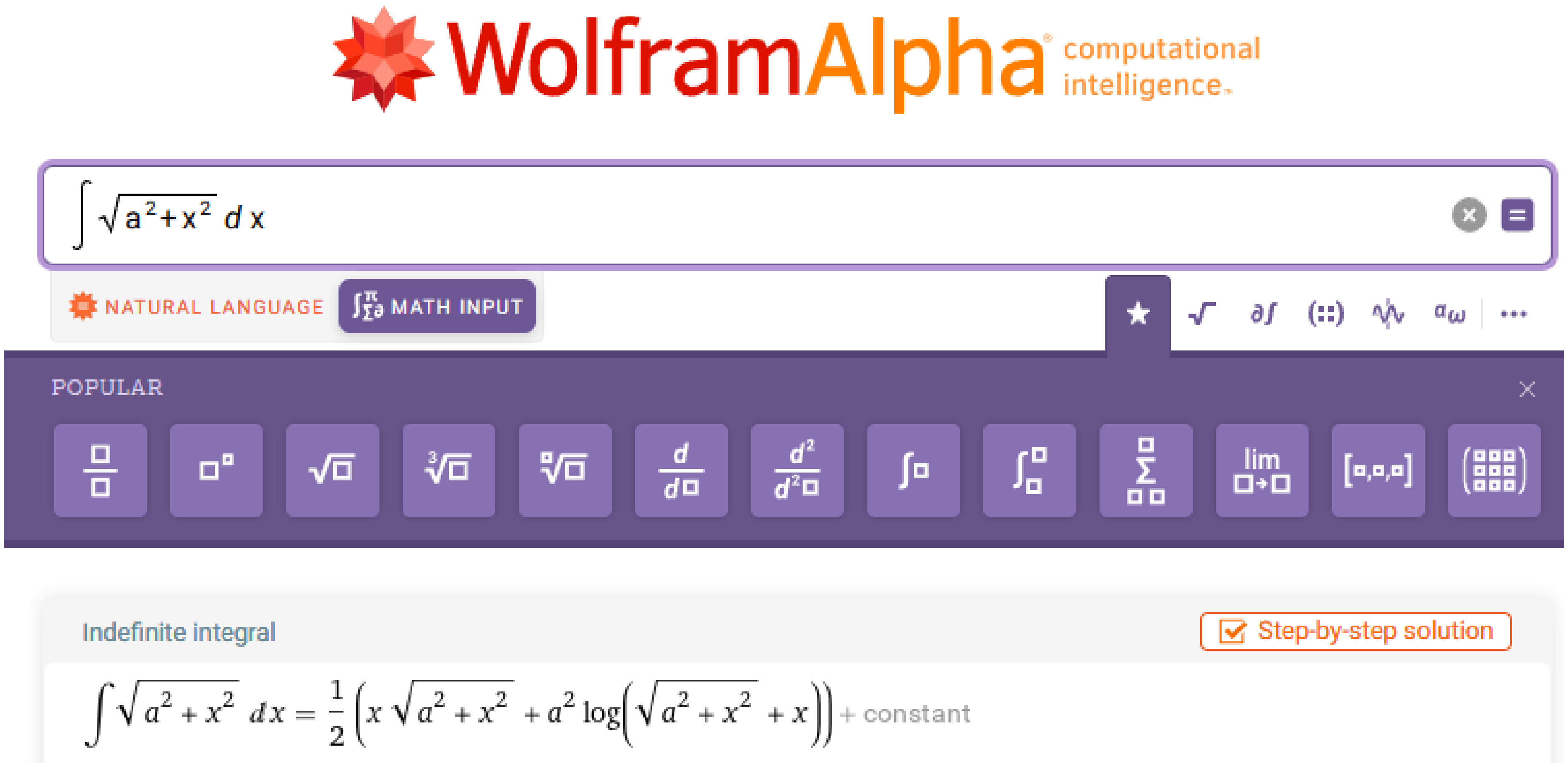
$$L = \int_0^2 \sqrt{1+(6x)^2}\,dx = 6\int_0^2 \sqrt{(1/6)^2 + x^2}\,dx$$

$$= \left\{6(1/2)\left[x\sqrt{x^2+(1/6)^2} + (1/6)^2 \log\left(x + \sqrt{x^2+(1/6)^2}\right)\right]\right\}_0^2$$

$$= 3\left\{2\sqrt{4+1/36} + \frac{1}{36}\left[\log\left(2+\sqrt{4+1/36}\right) - \log(1/6)\right]\right\}$$

$$= \sqrt{145} + \frac{1}{12}\log\left(12+\sqrt{145}\right)$$

Note: Use Wolfram Alpha for the indefinite integral



Example: Length of an ellipse

Find the arc length of the circumference of an ellipse given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

(Solution)

$$x=a\cos t,\ y=b\sin t\quad (0\le t\le 2\pi)$$

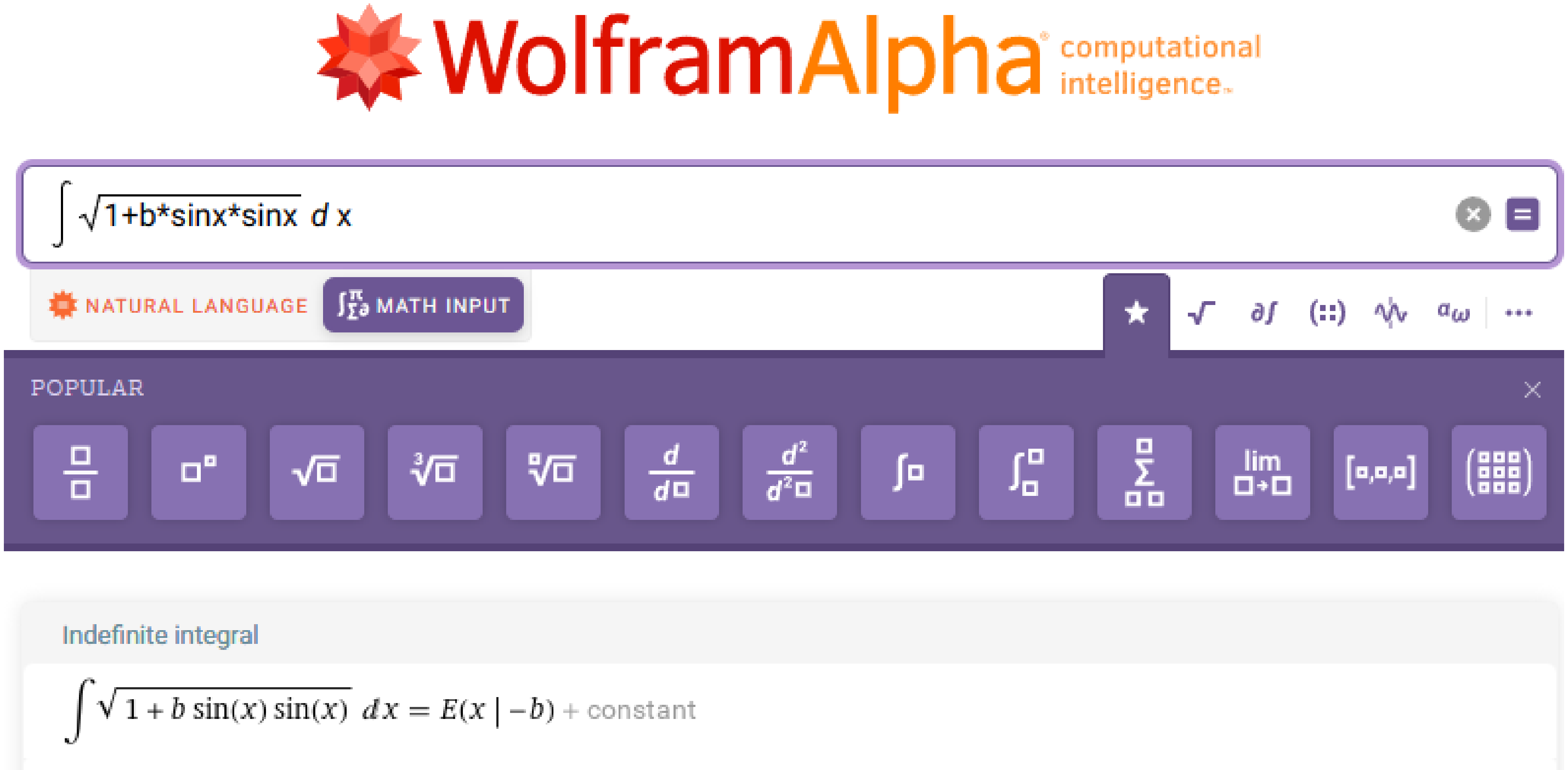
$$\frac{dx}{dt}=-a\sin t,\ \frac{dy}{dt}=b\cos t$$

$$L=\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^{2\pi}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,dt=\int_0^{2\pi}\sqrt{b^2+(a^2-b^2)\sin^2 t}\ dt$$

$$=b\int_0^{2\pi}\sqrt{1+\frac{a^2-b^2}{b^2}\sin^2 t}\ dt=4b\int_0^{\pi/2}\sqrt{1+\frac{a^2-b^2}{b^2}\sin^2 t}\ dt=4bE(-m)$$

$$m=\frac{a^2-b^2}{b^2}\ ;\quad E=\ \text{complete elliptic integral}$$

Note: Use Wolfram Alpha for the definite integral



2. Polar Coordinae System (극좌표계)

Coordinate: $P(\rho,\varphi)$ $(0 \le \rho < \infty,\ 0 \le \varphi \le 2\pi)$

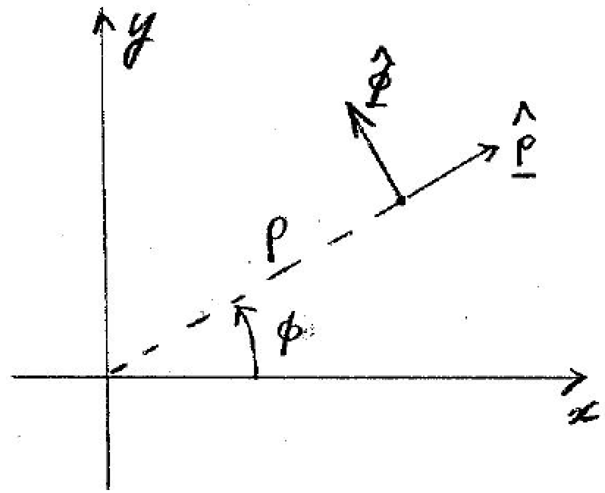
$$P(\rho,\varphi) = P(x, y)$$

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi$$

Base vectors: $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\varphi}}$

$$\hat{\boldsymbol{\rho}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$$

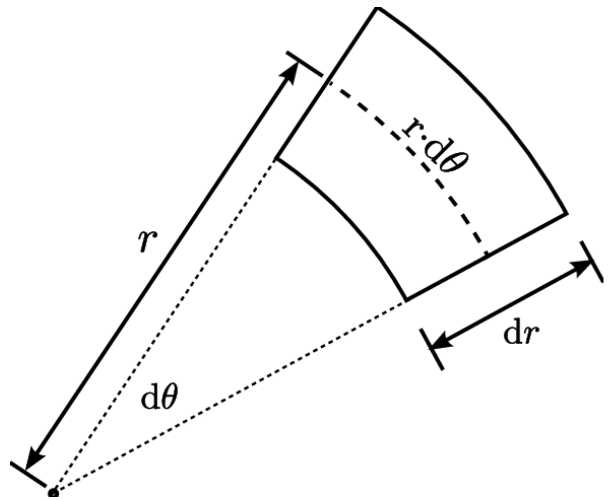


Differentials:

$$dL = \sqrt{(d\rho)^2 + (\rho d\varphi)^2}$$

$$d\mathbf{L} = (d\rho)\hat{\boldsymbol{\rho}} + (\rho d\varphi)\hat{\boldsymbol{\varphi}}$$

$$dS = \rho d\rho d\varphi$$



Distance:

$$R \neq \sqrt{(\rho_2 - \rho_1)^2 + (\varphi_2 - \varphi_1)^2},\ R = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$x_i = \rho_i \cos\varphi_i,\ y_i = \rho_i \sin\varphi_i \quad (i = 1, 2)$$

Vector field: $F(\rho,\varphi) = F_\rho(\rho,\varphi)\hat{\mathbf{x}} + F_\varphi(\rho,\varphi)\hat{\mathbf{y}}$

Example:

Convert $P(x, y) = (1, -2)$ into the polar coordinate $P(\rho, \varphi)$.

(Solution)

$$\rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-2)^2} = \sqrt{5}\ ;\ \varphi = \tan^{-1}\frac{-2}{\sqrt{5}} = -63.4°$$

Example:

At $P(x, y) = (1, -2)$, express $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}$

(Solution)

$$\rho = \sqrt{5},\ \varphi = -63.4°$$

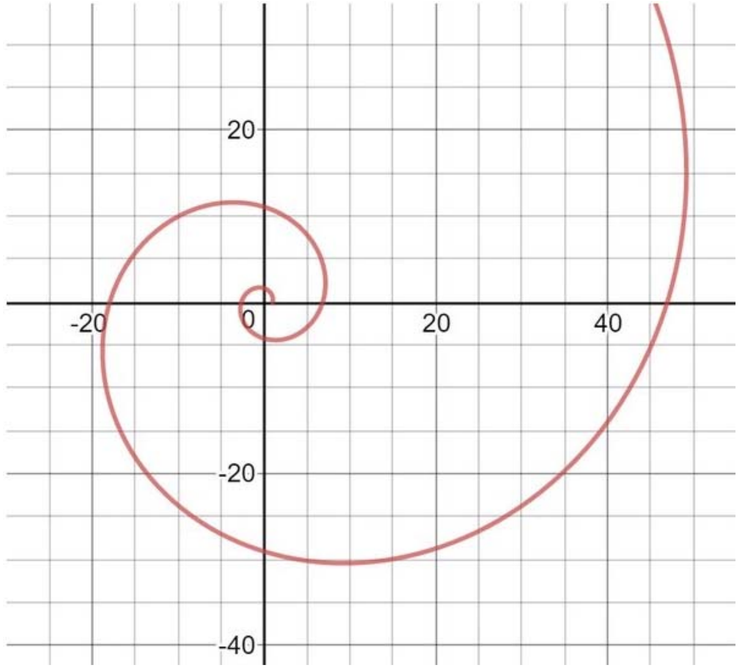
$$\hat{\boldsymbol{\rho}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}} = 0.448\hat{\mathbf{x}} - 0.894\hat{\mathbf{y}},\ \hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}} = 0.894\hat{\mathbf{x}} + 0.448\hat{\mathbf{y}}$$

Example: Length of a logarithmic spiral curve

Find the arc length of a curve $\rho = 3\exp(\varphi/\pi)$ for φ from 0 to 2π.

(Solution)

$$r = ae^{b\theta}$$



$$dL = \sqrt{(d\rho)^2 + (\rho d\varphi)^2} = \sqrt{\rho^2 + \left(\frac{d\rho}{d\varphi}\right)^2}\, d\varphi$$

$$\rho = 3e^{\varphi/\pi}$$

$$d\rho = (3/\pi)e^{\varphi/\pi} d\varphi$$

$$\frac{d\rho}{d\varphi} = \rho/\pi$$

$$dL = \rho\sqrt{1 + 1/\pi^2}\, d\varphi$$

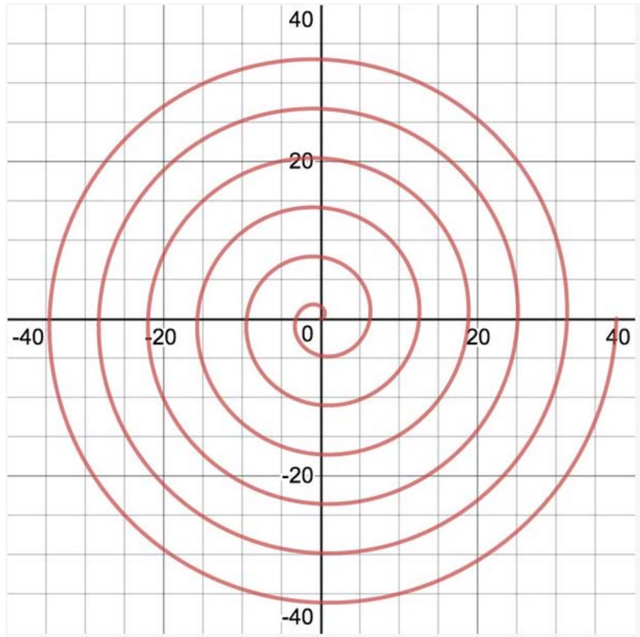
$$L = \int_0^{2\pi} 3e^{\varphi/\pi}\sqrt{1 + 1/\pi^2}\, d\varphi = (3\pi)\sqrt{1 + 1/\pi^2}\left[e^{\varphi/\pi}\right]_0^{2\pi} = 3(e^2 - 1)\sqrt{\pi^2 + 1}$$

Example: Length of an Archimedial or linear spiral curve

Find the arc length of a curve $\rho = \varphi$ for φ from φ_1 *to* φ_2.

$r = a + b\theta.$

(Solution)



$$dL = \sqrt{(d\rho)^2 + (\rho d\varphi)^2} = \sqrt{\rho^2 + \left(\frac{d\rho}{d\varphi}\right)^2}\, d\varphi$$

$$\rho = a\varphi$$

$$d\rho = a\, d\varphi$$

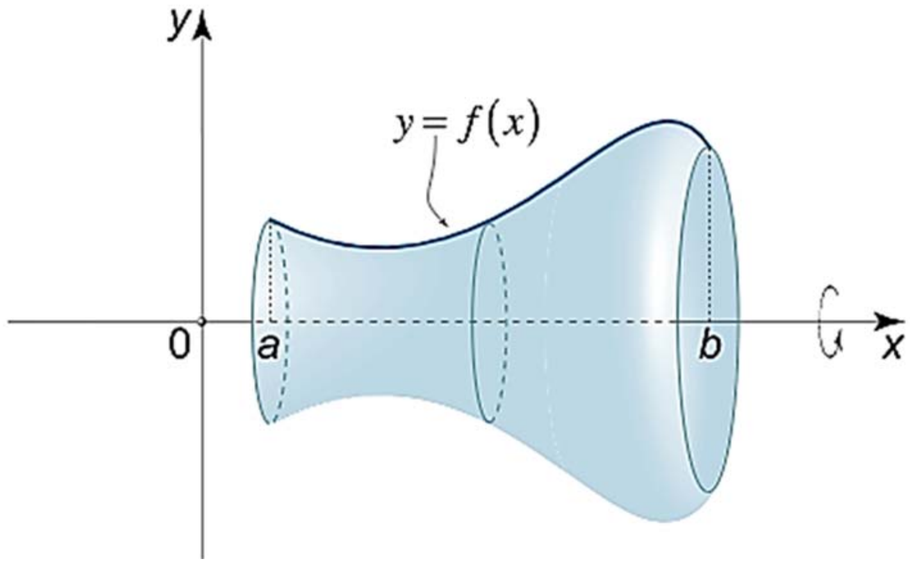
$$dL = a\sqrt{\varphi^2 + 1}\, d\varphi$$

$$L = a\int_{\varphi_1}^{\varphi_2} \sqrt{\varphi^2+1}\, d\varphi = a\left\{\frac{1}{2}\left[\varphi + \log\left(\varphi + \sqrt{\varphi^2+1}\right)\right]\right\}_{\varphi 1}^{\varphi_2} = \frac{a}{2}\left(\varphi_2 - \varphi_1 + \log\frac{\varphi_2 + \sqrt{\varphi_2^2+1}}{\varphi_1 + \sqrt{\varphi_1^2+1}}\right)$$

- **Surface area and volume of body of revolution (BOR)**

$$A = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx.$$

$$V = \pi \int_a^b f(x)^2 \, dx .$$



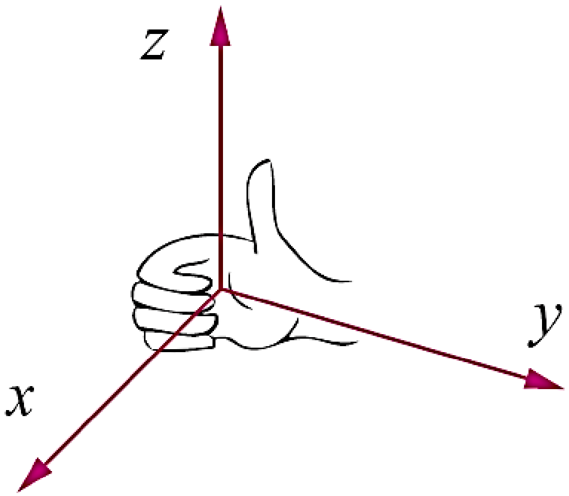
3. Rectangular Coordinate System (직각좌표계)

- Also called Cartesian coordinate system

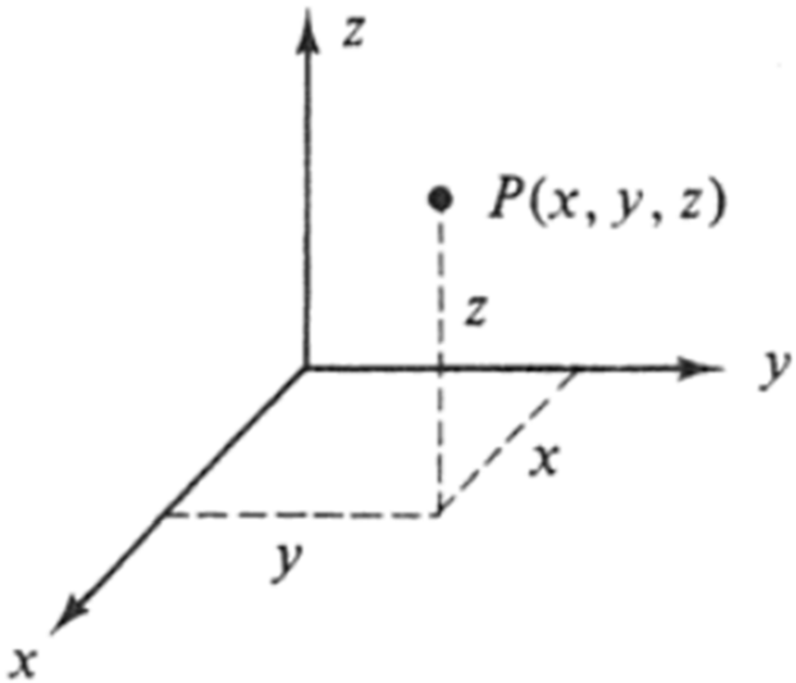
- Named after René Descarte(르네'이 데이카'알트)(1596-1650)

▪ Coordinate axes

- Right-hand rule: ordered triplet $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$



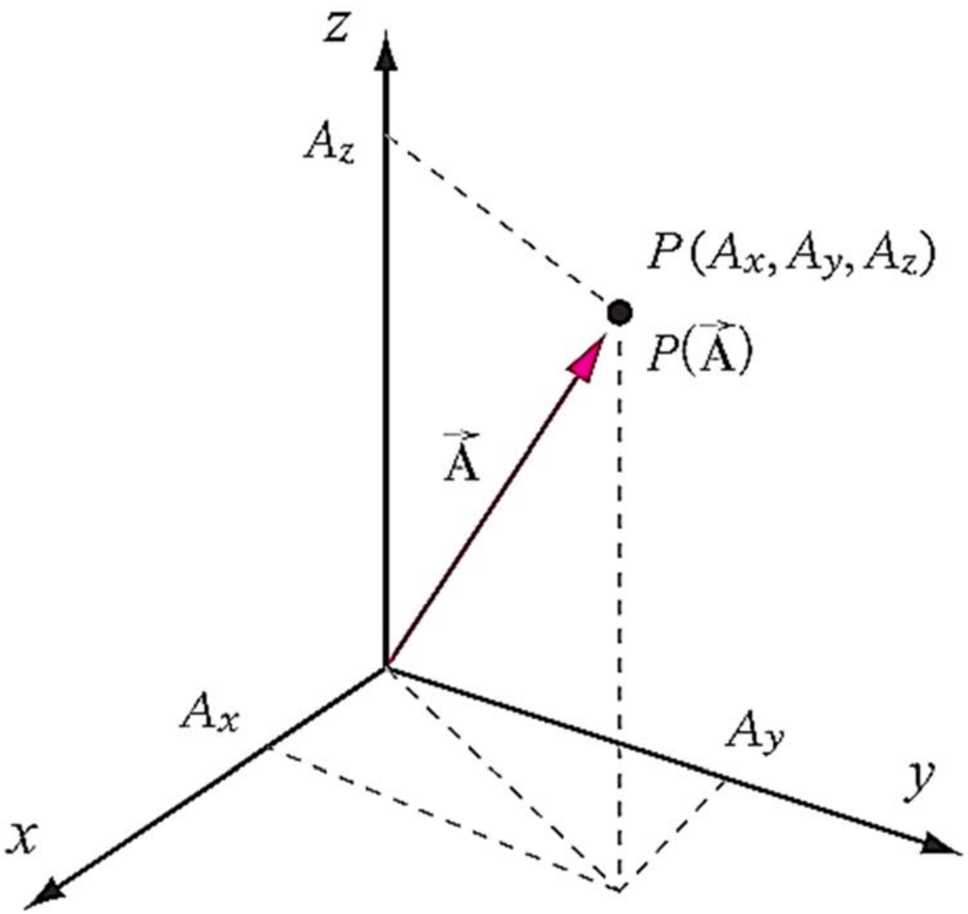
- **Coordinate of a point: (*x*, *y*, *z*)**



▪ Position vector

- Start point: (0, 0, 0)

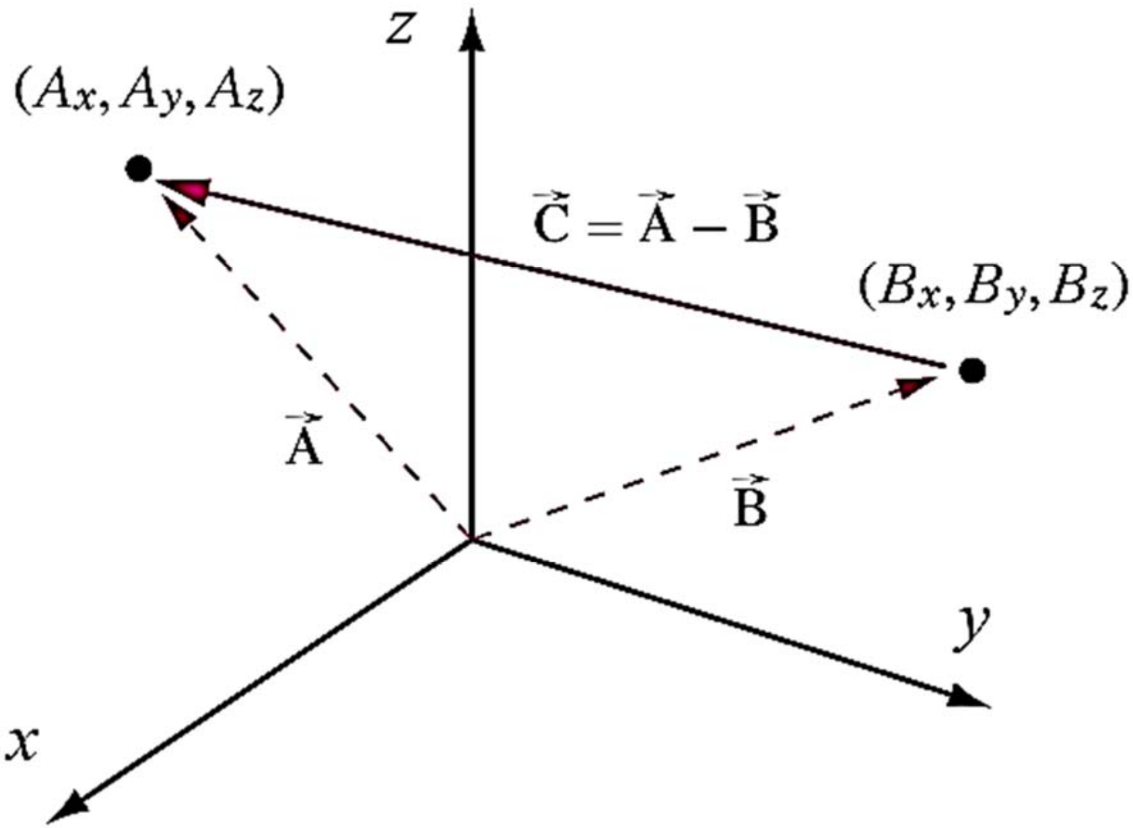
- End point: (A_x, A_y, A_z)



$$\vec{\mathbf{A}} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

- **Distance vector**

- A vector from a point *A* to a point *B*.



$$\vec{\mathbf{C}} = \vec{\mathbf{A}} - \vec{\mathbf{B}} = (A_x - B_x)\hat{\mathbf{x}} + (A_y - B_y)\hat{\mathbf{y}} + (A_z - B_z)\hat{\mathbf{z}}$$

Example:

직교좌표 (3,2,1) 을 시작점, (3, − 2, − 1)을 끝점으로 하는,

a) 변위벡터 $\vec{\mathbf{C}}$를 구하라.

b) 두 점을 잇는 선의 길이를 구하라.

(Solution)

a) $\vec{\mathbf{C}} = (3 - 3)\hat{\mathbf{x}} + (-2 - 2)\hat{\mathbf{y}} + (-1 - 1)\hat{\mathbf{z}} = -4\hat{\mathbf{y}} - 2\hat{\mathbf{z}}.$

b) 선의 길이는 $|\vec{\mathbf{C}}| = \sqrt{(-4)^2 + (-2)^2} = 2\sqrt{5}.$

Example:

(3, – 4, – 2)에서 (8, – 5,5)를 향하는 단위벡터 $\hat{C}$를 구하라.

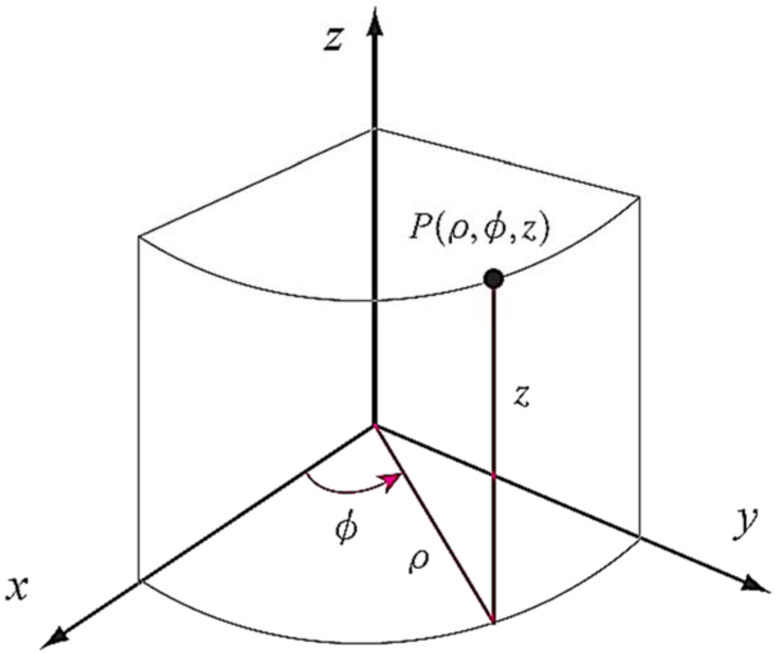
$$\vec{C} = (8 - 3)\hat{x} + (-5 + 4)\hat{y} + (5 + 2)\hat{z} = 5\hat{x} - \hat{y} + 7\hat{z}$$

(Solution)

$$\hat{C} = \frac{\vec{C}}{C} = \frac{5\hat{x} - \hat{y} + 7\hat{z}}{\sqrt{5^2 + (-1)^2 + 7^2}} = \frac{5\hat{x} - \hat{y} + 7\hat{z}}{5\sqrt{3}}.$$

4. Cylindrical Coordinate System (원통좌표계)

(ρ, ϕ, z) 로 좌표를 표시



ρ: z축으로부터의 거리

ϕ: $+x$축으로부터 반시계 방향의 각도

z: $x-y$평면으로부터의 높이

ρ = 뤄오우

ϕ = f아이

z = 지이

$\rho \geq 0$

$0 \leq \varphi \leq 2\pi$ or $-\pi \leq \varphi \leq \pi$

$-\infty \leq z \leq +\infty$

▪ Greek aphabet

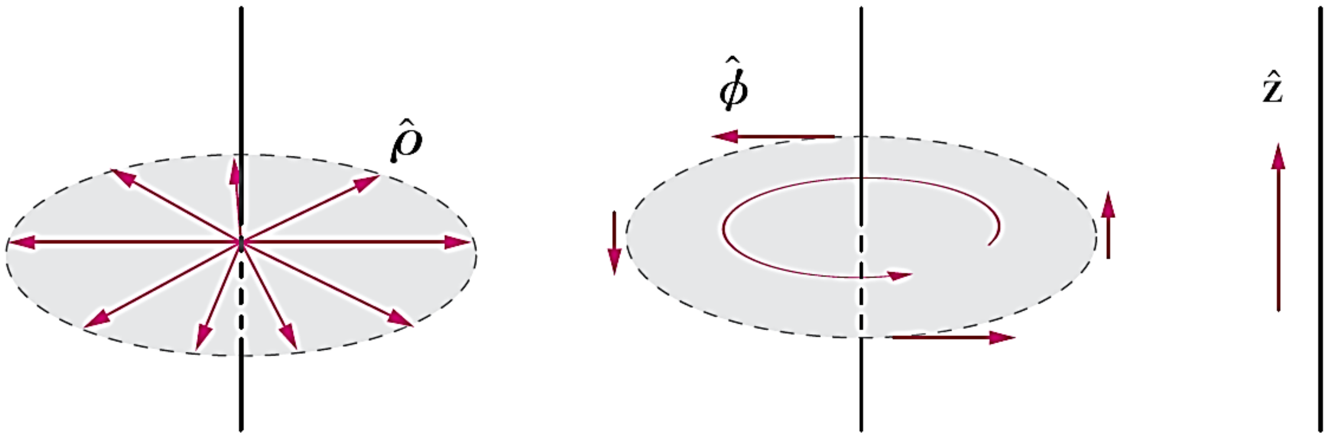
Α α	alpha	Ν ν	nu
Β β	beta	Ξ ξ	ksi
Γ γ	gamma	Ο ο	omicron
Δ δ	delta	Π π	pi
Ε ε	epsilon	Ρ ρ	rho
Ζ ζ	zeta	Σ σς	sigma
Η η	eta	Τ τ	tau
Θ θ	theta	Υ υ	upsilon
Ι ι	iota	Φ φ	phi
Κ κ	kappa	Χ χ	chi
Λ λ	lambda	Ψ ψ	psi
Μ μ	mu	Ω ω	omega

- 총 24 글자: 대문자, 소문자
- English alphabet: 26 elements
- 한글: 28 elements

epsilon = 엡'썰런
eta = 이'더
iota = 이오다, 아이오다
ξ = xi = ksi = 자이, 싸이, 크사이
omicron = 오'미크론
psi = 싸이,프사이
upsilon = 업'씰런
chi = 카이
psi = 프사이, 싸이
omega =

1. Exploring physics: Greek letters, https://www.youtube.com/watch?v=bjWux8GH4GY (3:58)
2. Greek alphabet song: https://www.youtube.com/watch?v=ZUrZHF_WBeI (2:12)

- **Base vectors:** $\hat{\rho}, \hat{\phi}, \hat{z}$

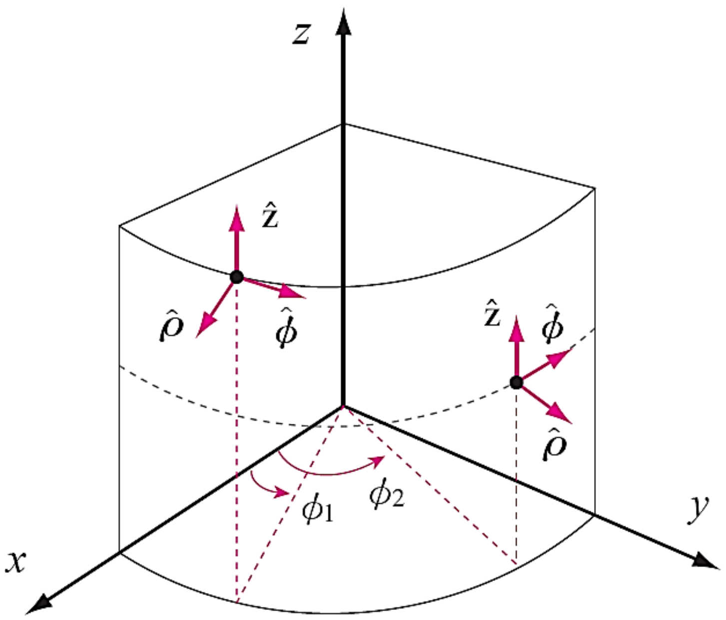


$\hat{\rho}$: ϕ, z는 변하지 않고 ρ만 증가하는 방향

$\hat{\phi}$: ρ, z는 변하지 않고 ϕ만 증가하는 방향

$\hat{z}$: ρ, ϕ는 변하지 않고 z만 증가하는 방향

- 원통좌표계의 기본단위벡터 $\hat{\rho}, \hat{\phi}, \hat{\mathbf{z}}$



$\hat{\rho}, \hat{\phi}, \hat{\mathbf{z}}$는 서로 수직

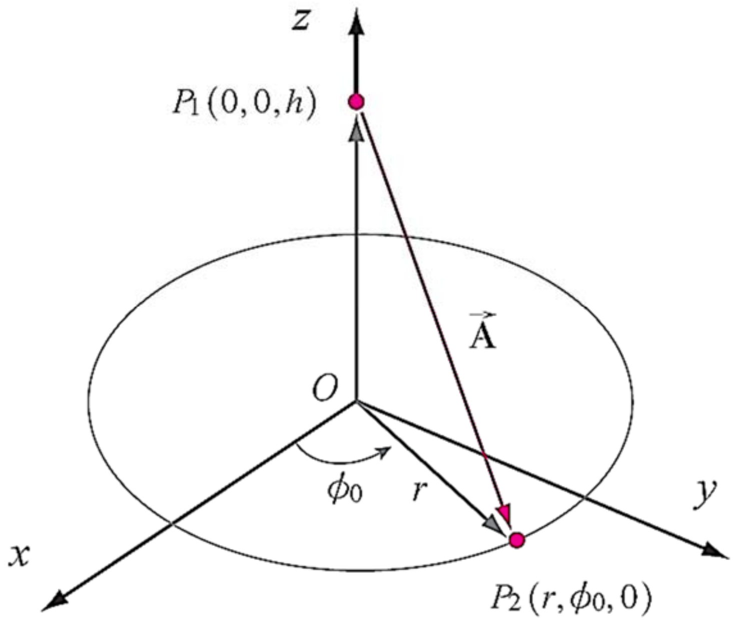
$\hat{\rho}, \hat{\phi}$는 ϕ좌표의 함수

원통좌표계의 기본단위벡터를 이용하여 각 방향성분의 크기가 (A_ρ, A_ϕ, A_z)인 벡터를 표현

$$\vec{\mathbf{A}} = A_\rho \hat{\boldsymbol{\rho}} + A_\phi \hat{\boldsymbol{\phi}} + A_z \hat{\mathbf{z}}$$

Example:

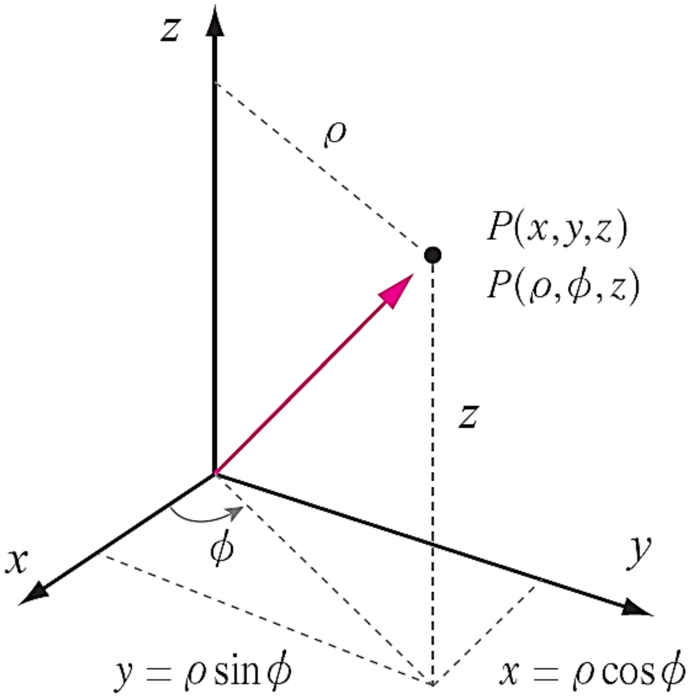
그림 E1-13의 $\vec{A}$를 원통좌표계 기본단위벡터의 조합으로 표현하라.



(Solution)

$$\vec{\mathbf{A}} = \vec{\mathbf{P}}_2 - \vec{\mathbf{P}}_1 = r\hat{\boldsymbol{\rho}} - h\hat{\mathbf{z}} \quad (\phi = \phi_0)$$

▪ 직교좌표 ↔ 원통좌표 변환



직교좌표를 원통좌표로

또는 그 반대로 변환할 수 있음

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

↔

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

Example:

$P(2, 2, 2)$를 원통좌표로 변환하라.

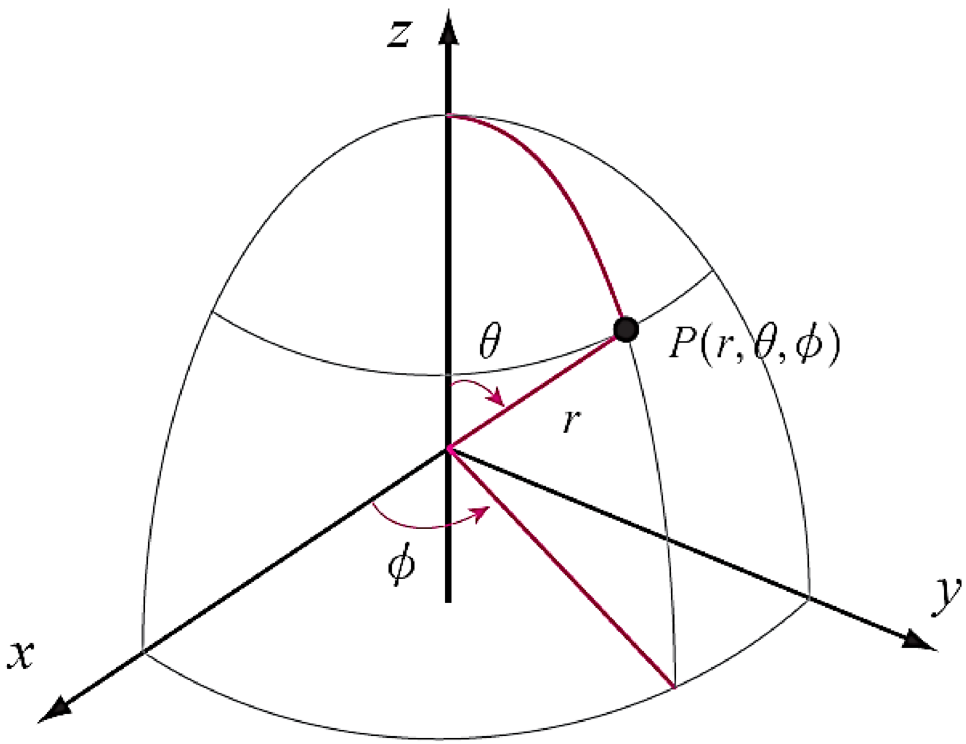
(Solution)

$$\rho = \sqrt{2^2 + 2^2} = 2\sqrt{2}\ ,\quad \phi = \tan^{-1}(2/2) = 45^\circ\ ,\quad z = 2$$

$$P\left(2\sqrt{2}, 45^\circ, 2\right)$$

5. Spherical Coordinate System (구좌표계)

(r,θ,ϕ) 로 좌표를 표시



r: 원점으로부터의 거리

θ: $+z$ 축으로부터의 각

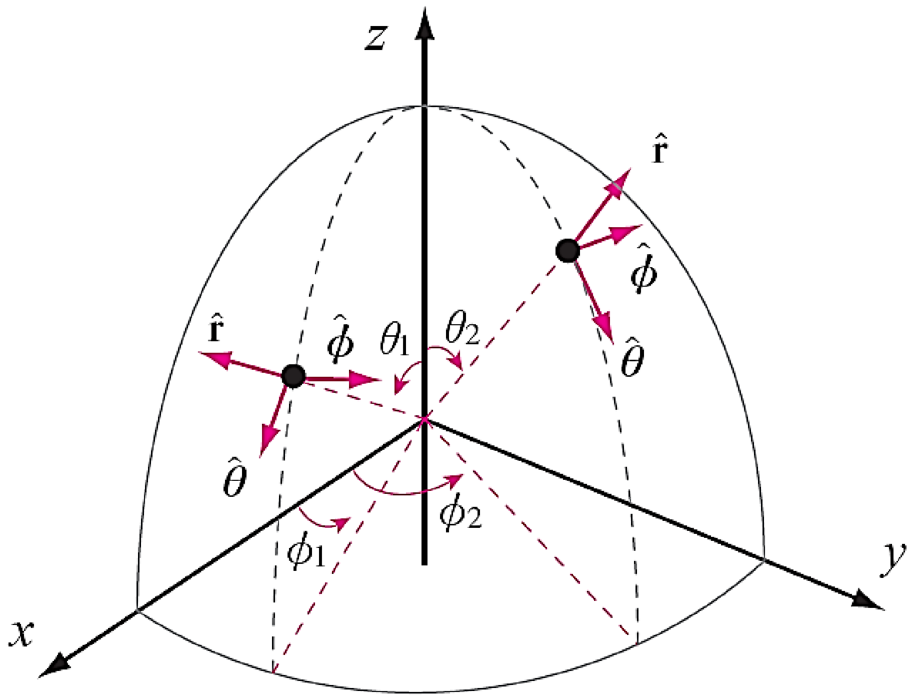
ϕ: $+x$ 축으로부터 반시계 방향의 각

$$r \geq 0$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \varphi \leq 2\pi \quad \text{or} \quad -\pi \leq \varphi \leq \pi$$

- 구좌표계의 기본단위벡터 $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$



$\hat{\mathbf{r}}$: θ, ϕ는 변하지 않고 r만 증가하는 방향

$\hat{\theta}$: r, ϕ는 변하지 않고 θ만 증가하는 방향

$\hat{\phi}$: r, θ는 변하지 않고 ϕ만 증가하는 방향

$\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$ 는 서로 수직

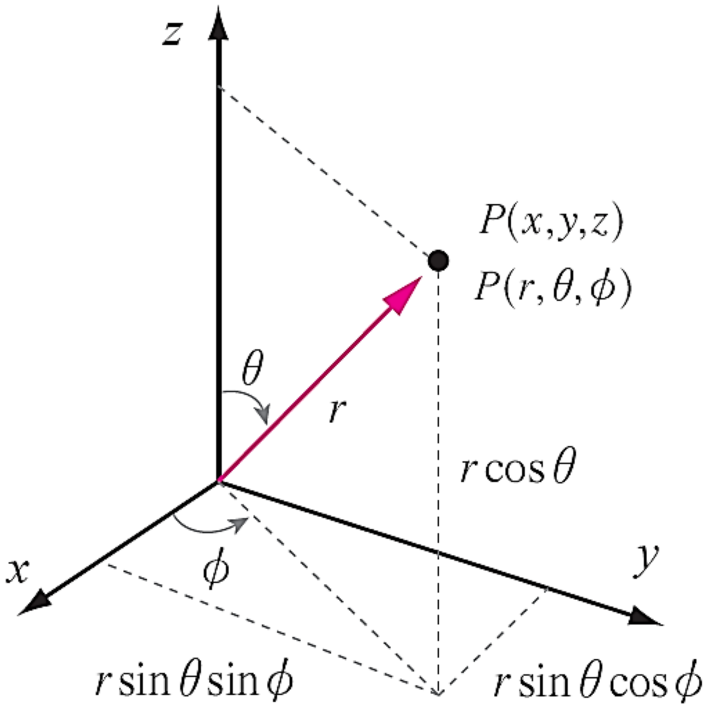
$\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$ 는 ϕ의 함수

$\hat{\mathbf{r}}, \hat{\theta}$ 는 θ의 함수

구좌표계의 기본단위벡터를 이용하여 각 방향성분의 크기가 (A_r, A_θ, A_ϕ) 인 벡터를 표현

$$\vec{\mathbf{A}} = A_r \hat{\mathbf{r}} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$$

- 직교좌표 ↔ 구좌표 변환



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

↔

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}(\sqrt{x^2 + y^2}/z)$$
$$\phi = \tan^{-1}(y/x)$$

Example:

구좌표 $P(4, 60°, 300°)$의 위치를 표현하는 직교좌표계의 위치벡터를 구하라.

(Solution)

$$x = r\sin\theta\cos\phi \qquad x = 4\sin 60^\circ \cos 300^\circ = \sqrt{3}$$

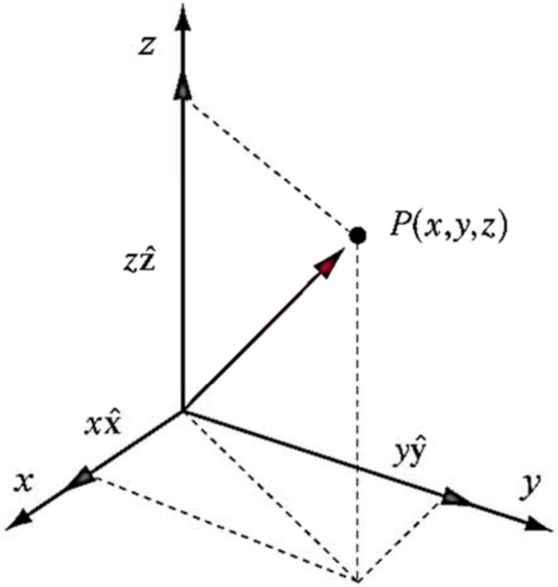
$$y = r\sin\theta\sin\phi \quad \longrightarrow \quad y = 4\sin 60^\circ \sin 300^\circ = -3$$

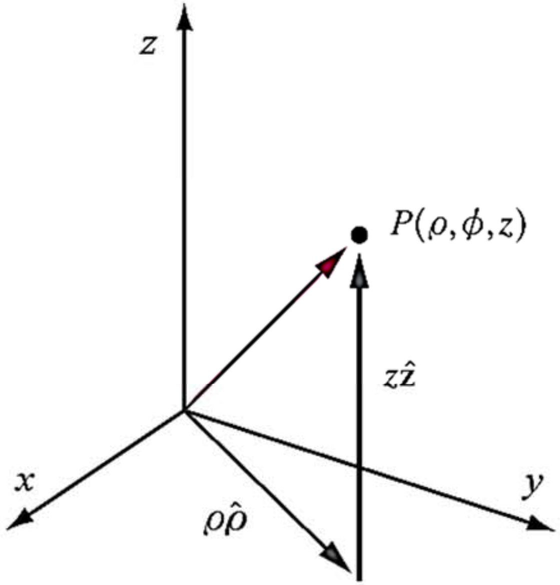
$$z = r\cos\theta \qquad z = 4\cos 60^\circ = 2$$

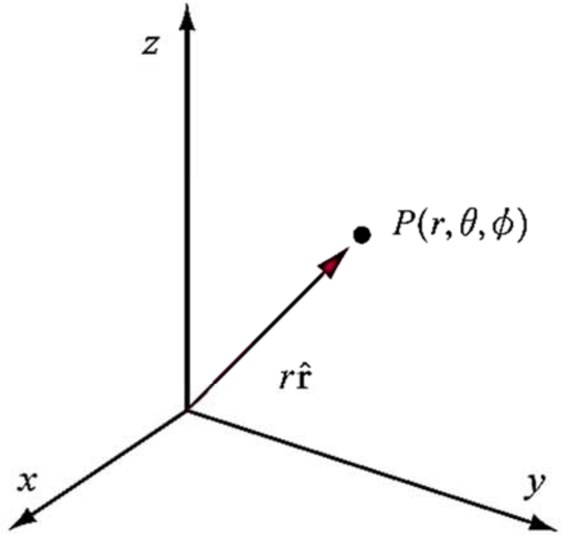
$$\downarrow$$

$$P(\sqrt{3}, -3, 2) \qquad \vec{\mathbf{P}} = \sqrt{3}\,\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$$

- 각 좌표계의 위치벡터 표현







$$\vec{\mathbf{P}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$\vec{\mathbf{P}} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

$$\vec{\mathbf{P}} = r\hat{\mathbf{r}}$$

상황에 맞추어서 계산과 표현을 쉽게 할 수 있는 좌표계를 사용

6. Coordinate Transform (좌표계 변환)

▪ 직각좌표계 ↔ 원통좌표계 변환

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \longrightarrow \vec{A} = A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}$$

- 전자기장 문제 해석에 있어서 좌표계 기본벡터 변환 필요

$\hat{x}, \hat{y}, \hat{z} \longleftrightarrow \hat{\rho}, \hat{\phi}, \hat{z}$

$\hat{x}, \hat{y}, \hat{z}$ 각각을 단위벡터 $\hat{\rho}, \hat{\phi}, \hat{z}$ 로 표현한다.

$$\hat{\mathbf{x}} = a\hat{\boldsymbol{\rho}} + b\hat{\boldsymbol{\varphi}} + c\hat{\mathbf{z}} = \cos\varphi\hat{\boldsymbol{\rho}} - \sin\varphi\hat{\boldsymbol{\varphi}}$$

a : $\hat{\mathbf{x}}$ 의 $\hat{\boldsymbol{\rho}}$ 방향 성분의 크기 $\rightarrow a = \hat{\mathbf{x}} \cdot \hat{\boldsymbol{\rho}} = \cos\varphi$, 두 벡터의 사이각 = ϕ

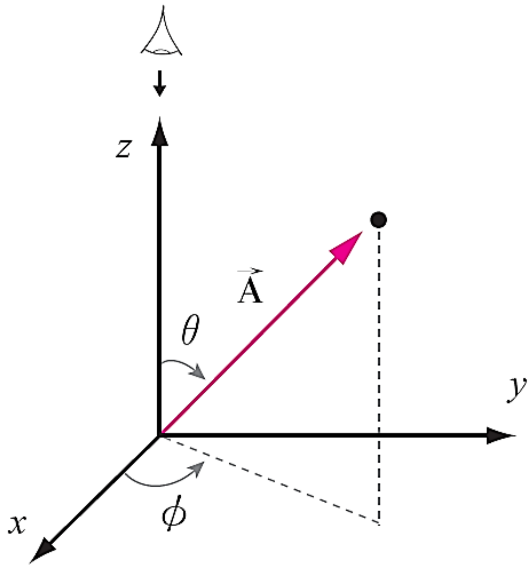
b : $\hat{\mathbf{x}}$ 의 $\hat{\boldsymbol{\varphi}}$ 방향 성분의 크기 $\rightarrow b = \hat{\mathbf{x}} \cdot \hat{\boldsymbol{\varphi}} = \cos(\varphi + \pi/2) = -\sin\varphi$

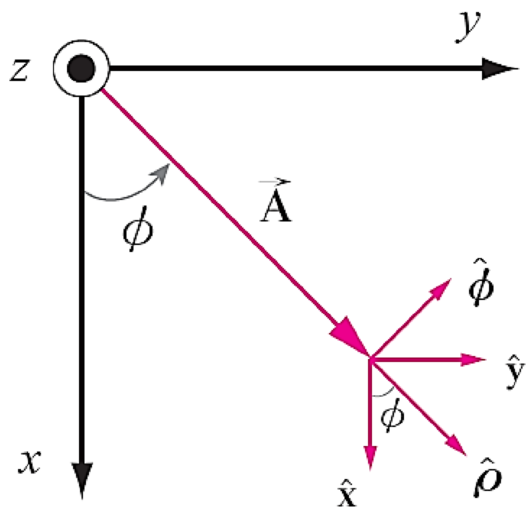
두 벡터의 사이각 = $\varphi + \pi/2$

c : $\hat{\mathbf{x}}$ 의 $\hat{\mathbf{z}}$ 방향 성분의 크기 → $c = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \cos 90° = 0$

두 벡터의 사이각 = 90°

▪ 기본단위벡터 변환공식(직교↔원통)





$$\hat{x} = a\hat{\rho} + b\hat{\phi}$$
$$\hat{y} = c\hat{\rho} + d\hat{\phi}$$

→

$$\hat{x}\cdot\hat{\rho} = (a\hat{\rho} + b\hat{\phi})\cdot\hat{\rho} = a = \cos\phi$$
$$\hat{x}\cdot\hat{\phi} = (a\hat{\rho} + b\hat{\phi})\cdot\hat{\phi} = b = -\sin\phi$$
$$\hat{y}\cdot\hat{\rho} = (c\hat{\rho} + d\hat{\phi})\cdot\hat{\rho} = c = \sin\phi$$
$$\hat{y}\cdot\hat{\phi} = (c\hat{\rho} + d\hat{\phi})\cdot\hat{\phi} = d = \cos\phi$$

→

$$\hat{x} = \cos\phi\hat{\rho} - \sin\phi\hat{\phi}$$
$$\hat{y} = \sin\phi\hat{\rho} + \cos\phi\hat{\phi}$$

– 같은 방법으로 다음 식을 구한다.

$$\hat{\rho} = \cos\phi\hat{x} + \sin\phi\hat{y}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

	$\hat{\rho}$	$\hat{\varphi}$	$\hat{z}$
$\hat{x}$	φ	$\pi/2+\varphi$	$\pi/2$
$\hat{y}$	$\pi/2-\varphi$	φ	$\pi/2$
$\hat{z}$	$\pi/2$	$\pi/2$	0

- 기본단위벡터 변환공식 (원통↔직교)

$$\hat{\mathbf{x}} = \cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{y}} = \sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}}$$

$$\hat{\boldsymbol{\rho}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

$$\vec{\mathrm{A}} = A_\rho\hat{\boldsymbol{\rho}} + A_\phi\hat{\boldsymbol{\phi}} + A_z\hat{\mathbf{z}}$$

$$\vec{\mathrm{A}} = (A_\rho\cos\phi - A_\phi\sin\phi)\hat{\mathbf{x}} + (A_\rho\sin\phi + A_\phi\cos\phi)\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}} = A_x(\cos\varphi\hat{\boldsymbol{\rho}} - \sin\varphi\hat{\boldsymbol{\varphi}}) + A_y(\sin\varphi\hat{\boldsymbol{\rho}} + \cos\varphi\hat{\boldsymbol{\varphi}}) + A_z\hat{\mathbf{z}}$$

$$= (A_x\cos\varphi + A_y\sin\varphi)\hat{\boldsymbol{\rho}} + (-A_x\sin\varphi + A_y\cos\varphi)\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$$

Example:

$\vec{A} = 2\hat{x} + 2\hat{y} + 2\hat{z}$를 원통좌표계의 벡터로 변환하라.

(Solution)

$$\vec{A} = 2\hat{x} + 2\hat{y} + 2\hat{z}$$

$$\hat{x} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}$$
$$\hat{y} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}$$

$$\vec{A} = (2\cos\phi + 2\sin\phi)\,\hat{\rho} + (-2\sin\phi + 2\cos\phi)\,\hat{\phi} + 2\hat{z}$$

$$\phi = \tan^{-1}(2/2) = 45°$$

$$\vec{A} = 2\sqrt{2}\,\hat{\rho} + 2\hat{z}$$

Example:

$\vec{\mathbf{A}} = y\hat{\mathbf{x}} + x\hat{\mathbf{y}} + \dfrac{x^2}{\sqrt{x^2+y^2}}\hat{\mathbf{z}}$을 원통좌표계의 벡터로 변환하라.

(Solution)

$$\vec{\mathbf{A}} = y\hat{\mathbf{x}} + x\hat{\mathbf{y}} + \frac{x^2}{\sqrt{x^2+y^2}}\hat{\mathbf{z}}$$

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

$$\vec{\mathbf{A}} = \rho(\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} + \cos^2\phi\hat{\mathbf{z}})$$

$$\hat{\mathbf{x}} = \cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{y}} = \sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}}$$

$$\vec{\mathbf{A}} = 2\rho\sin\phi\cos\phi\hat{\boldsymbol{\rho}} + \rho(\cos^2\phi - \sin^2\phi)\hat{\boldsymbol{\phi}} + r\cos^2\phi\hat{\mathbf{z}}$$

$$= 2\rho[\sin 2\phi\hat{\boldsymbol{\rho}} + (1 - 2\sin^2\phi)\hat{\boldsymbol{\phi}} + \cos^2\phi\hat{\mathbf{z}}].$$

Example:

$\vec{A} = 3\cos\phi\hat{\rho} - 2\rho\hat{\phi} + 5\hat{z}$를 직교좌표계의 벡터로 표현하라.

(Solution)

$$\vec{A} = 3\cos\phi\hat{\rho} - 2\rho\hat{\phi} + 5\hat{z}$$

$$\hat{\rho} = \cos\phi\hat{x} + \sin\phi\hat{y}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

$$\vec{A} = 3\cos\phi(\cos\phi\hat{x} + \sin\phi\hat{y}) - 2\rho(-\sin\phi\hat{x} + \cos\phi\hat{y}) + 5\hat{z}$$

$$= (3\cos^2\phi + 2\rho\sin\phi)\hat{x} + (3\cos\phi\sin\phi - 2\rho\cos\phi)\hat{y} + 5\hat{z}.$$

$$\cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\vec{A} = \left(\frac{3x^2}{x^2+y^2} + 2y\right)\hat{x} + \left(\frac{3xy}{x^2+y^2} - 2x\right)\hat{y} + 5\hat{z}$$

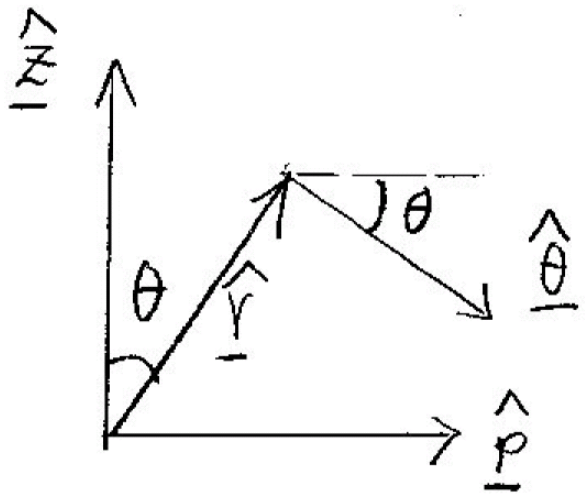
■ 직각좌표계 ↔ 구좌표계 변환

$\vec{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$ → $\vec{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$

직교좌표계의 벡터 구좌표계의 벡터

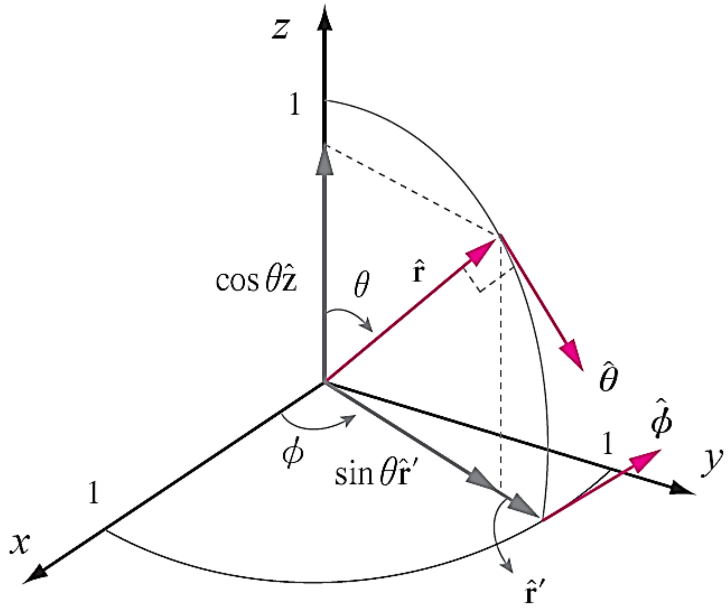
$\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ ↔ $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$

기본단위벡터간의 변환공식이 필요함



	$\hat{\mathbf{r}}$	$\hat{\boldsymbol{\theta}}$
$\hat{\boldsymbol{\rho}}$	$\pi/2-\theta$	θ
$\hat{\mathbf{z}}$	θ	$\pi/2+\theta$

- 기본단위벡터 변환공식(구좌표계→직각좌표계)



$$\hat{\mathbf{r}}' = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}} = \hat{\boldsymbol{\rho}}$$

$$\begin{aligned}\mathbf{r} &= \sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{z}} \\ &= \sin\theta(\cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}) + \cos\theta\,\hat{\mathbf{z}}\end{aligned}$$

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= \cos\theta\,\hat{\boldsymbol{\rho}} - \sin\theta\,\hat{\mathbf{z}} \\ &= \cos\theta(\cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}) - \sin\theta\,\hat{\mathbf{z}}\end{aligned}$$

$$\hat{\boldsymbol{\varphi}} = \cos(\pi/2 + \varphi)\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$

$$\vec{\mathbf{A}} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}} \longrightarrow$$

$$\begin{aligned}\vec{\mathbf{A}} = {} & (A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi)\,\hat{\mathbf{x}} \\ & + (A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi)\,\hat{\mathbf{y}} \\ & + (A_r\cos\phi - A_\theta\sin\theta)\,\hat{\mathbf{z}}\end{aligned}$$

- 기본단위벡터 변환공식 (직각좌표계→구좌표계)

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

→

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\theta\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

↓

$$\begin{aligned}\vec{\mathbf{A}} = {} & (A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta)\,\hat{\mathbf{r}} \\ & + (A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta)\,\hat{\boldsymbol{\theta}} \\ & + (A_y\cos\phi - A_x\sin\phi)\,\hat{\boldsymbol{\phi}}\end{aligned}$$

- 좌표와 벡터의 변환공식

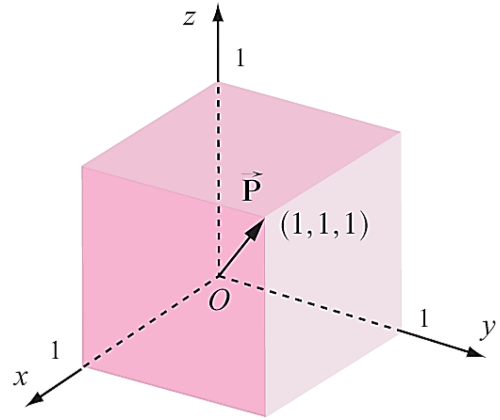
변환	좌표	벡터
직교→원통	$\rho = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$A_\rho = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
원통→직교	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$A_x = A_\rho \cos\phi - A_\phi \sin\phi$ $A_y = A_\rho \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
직교→구	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right)$ $\phi = \tan^{-1}(y/x)$	$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$ $A_\phi = A_y \cos\phi - A_x \sin\phi$
구→직교	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$A_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$ $A_z = A_r \cos\phi - A_\theta \sin\theta$
원통→구	$r = \sqrt{\rho^2 + z^2}$ $\theta = \tan^{-1}(\rho/z)$ $\phi = \phi$	$A_r = A_\rho \sin\theta + A_z \cos\theta$ $A_\theta = A_\rho \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
구→원통	$\rho = r\sin\theta$ $\phi = \phi$ $z = r\cos\theta$	$A_\rho = A_r \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_r \cos\theta - A_\theta \sin\theta$

Example:

그림 E1-20의 위치벡터 $\vec{\mathbf{P}}$를 벡터의 변환공식을 사용하여,

a) 원통좌표계의 벡터로 표현하라.

b) 구좌표계의 벡터로 표현하라.



(Solution)

a) $\vec{\mathbf{P}} = (\cos\phi + \sin\phi)\hat{\boldsymbol{\rho}} + (-\sin\phi + \cos\phi)\hat{\boldsymbol{\phi}} + \hat{\mathbf{z}}$ ← $\phi = \tan^{-1}(1) = \pi/4$

$\vec{\mathbf{P}} = \sqrt{2}\,\hat{\boldsymbol{\rho}} + \hat{\mathbf{z}}$

b) $$\vec{\mathbf{P}} = (\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\hat{\mathbf{r}} + (\cos\theta\cos\phi + \cos\theta\sin\phi - \sin\theta)\hat{\boldsymbol{\theta}} + (-\sin\phi + \cos\phi)\hat{\boldsymbol{\phi}}$$

$$= \cos\theta(\tan\theta\cos\phi + \tan\theta\sin\phi + 1)\hat{\mathbf{r}} + \cos\theta(\cos\phi + \sin\phi - \tan\theta)\hat{\boldsymbol{\theta}} + (-\sin\phi + \cos\phi)\hat{\boldsymbol{\phi}}.$$

$\tan\theta = \sqrt{x^2 + y^2}/z = \sqrt{2}$, $\phi = \tan^{-1}(1) = \pi/4$이므로,

$$\vec{\mathbf{P}} = 3\cos\theta\,\hat{\mathbf{r}}.$$

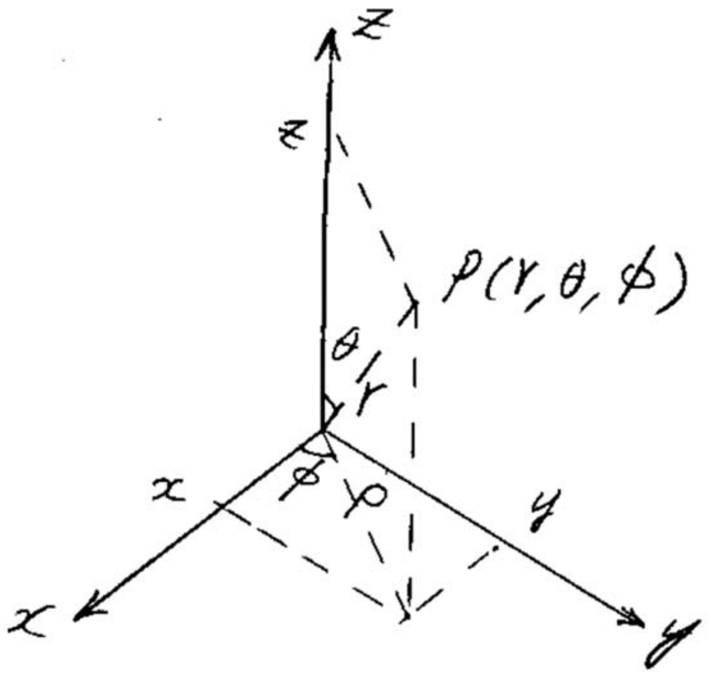
$P_r = \sqrt{3}$ 이므로 위 식과 비교하면 $\cos\theta = 1/\sqrt{3}$ → $\vec{\mathbf{P}} = \sqrt{3}\,\hat{\mathbf{r}}$

Example:

Drill: $P(r=3, \theta=60^\circ, \phi=30^\circ)$.

1) $P(x, y, z)$?

2) $(\hat{\underline{r}}, \hat{\underline{\theta}}, \hat{\underline{\phi}})$ at P ?



(Solution)

1)

$$z = r\cos\theta = 3\cos 60° = 3 \times \frac{1}{2} = \frac{3}{2}$$

$$\rho = r\sin\theta = 3\sin 60° = 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$x = \rho\cos\phi = \frac{3\sqrt{3}}{2}\cos 30° = \frac{3\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{9}{4}$$

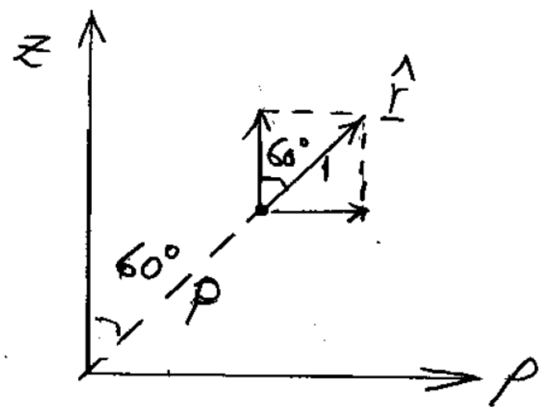
$$y = \rho\sin\phi = \frac{3\sqrt{3}}{2}\sin 30° = \frac{3\sqrt{3}}{2} \times \frac{1}{2} = \frac{3\sqrt{3}}{4}$$

$$P(x, y, z) = P\left(\frac{9}{4}, \frac{3\sqrt{3}}{4}, \frac{3}{2}\right)$$

Check: $\sqrt{x^2+y^2+z^2} \stackrel{?}{=} r$

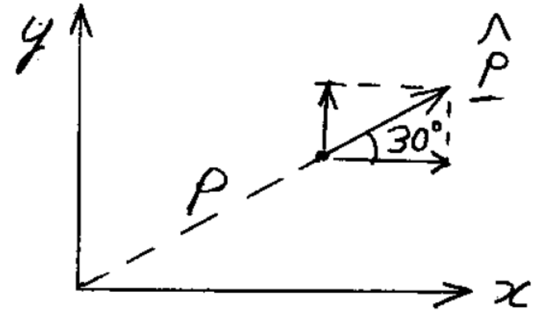
$$\sqrt{\frac{9^2}{16} + \frac{9\times 3}{16} + \frac{6^2}{16}} = \sqrt{\frac{164}{16}} = \sqrt{9} = 3 \quad \text{(Correct!)}$$

2)



$$\hat{\underline{r}} = \cos 60° \, \hat{\underline{z}} + \sin 60° \, \hat{\underline{\rho}}$$

$$= \frac{1}{2}\hat{\underline{z}} + \frac{\sqrt{3}}{2}\hat{\underline{\rho}}$$



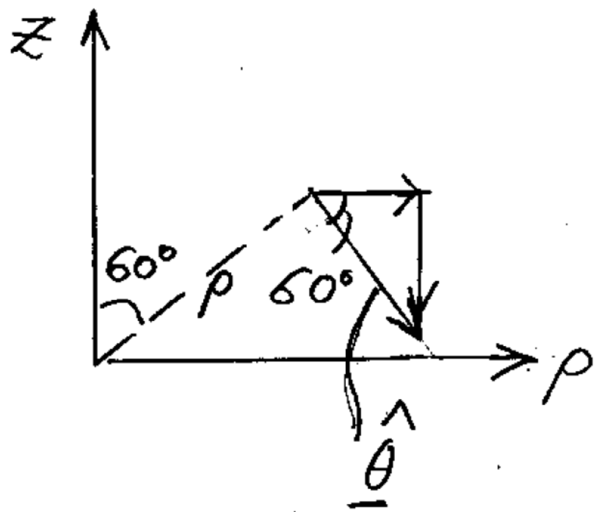
$$\hat{\underline{\rho}} = \cos 30° \, \hat{\underline{x}} + \sin 30° \, \hat{\underline{y}}$$

$$= \frac{\sqrt{3}}{2}\hat{\underline{x}} + \frac{1}{2}\hat{\underline{y}}$$

$$\hat{\underline{r}} = \frac{1}{2}\hat{\underline{z}} + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\hat{\underline{x}} + \frac{1}{2}\hat{\underline{y}}\right) = \frac{3}{4}\hat{\underline{x}} + \frac{\sqrt{3}}{4}\hat{\underline{y}} + \frac{1}{2}\hat{\underline{z}}$$

Check: $|\hat{\underline{r}}| \stackrel{?}{=} 1$

$$\sqrt{\frac{9}{16} + \frac{3}{16} + \frac{4}{16}} = \sqrt{1} = 1 \text{ (Correct!)}$$



$$\underline{\hat{\theta}} = \cos 60° \underline{\hat{\rho}} + \sin 60° (-\underline{\hat{z}})$$

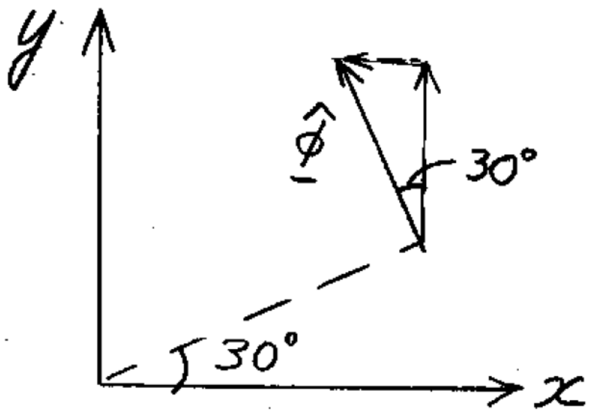
$$= \frac{1}{2}\underline{\hat{\rho}} - \frac{\sqrt{3}}{2}\underline{\hat{z}}$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\underline{\hat{x}} + \frac{1}{2}\underline{\hat{y}}\right) - \frac{\sqrt{3}}{2}\underline{\hat{z}}$$

$$= \frac{\sqrt{3}}{4}\underline{\hat{x}} + \frac{1}{4}\underline{\hat{y}} - \frac{\sqrt{3}}{2}\underline{\hat{z}}$$

Check: $\underline{\hat{r}} \perp \underline{\hat{\theta}} \rightarrow \underline{\hat{r}} \cdot \underline{\hat{\theta}} = 0$

$$\left(\frac{3}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right) \cdot \left(\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{16} + \frac{\sqrt{3}}{16} - \frac{\sqrt{3}}{4} = 0 \text{ (Correct!)}$$

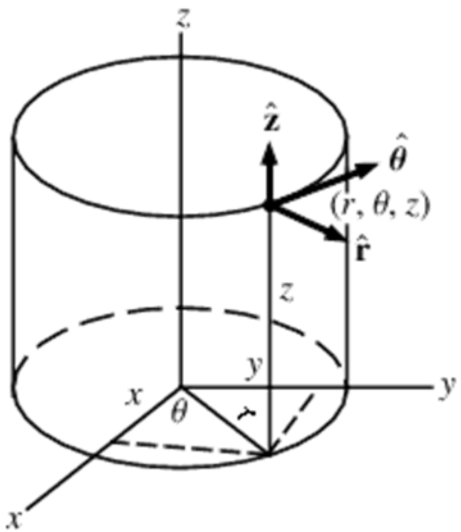


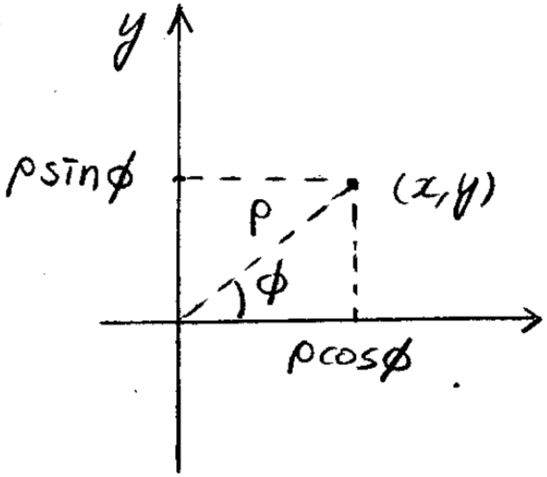
$$\hat{\underline{\phi}} = \cos 30° \hat{\underline{y}} + \sin 30° (-\hat{\underline{x}})$$

$$= -\frac{1}{2}\hat{\underline{x}} + \frac{\sqrt{3}}{2}\hat{\underline{y}}$$

Summary:

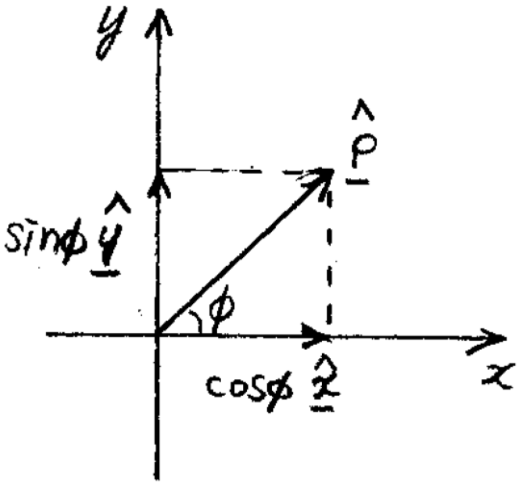
- 원통좌표계

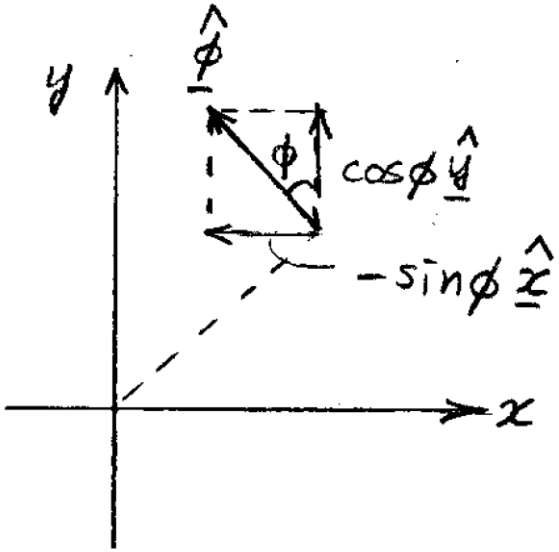




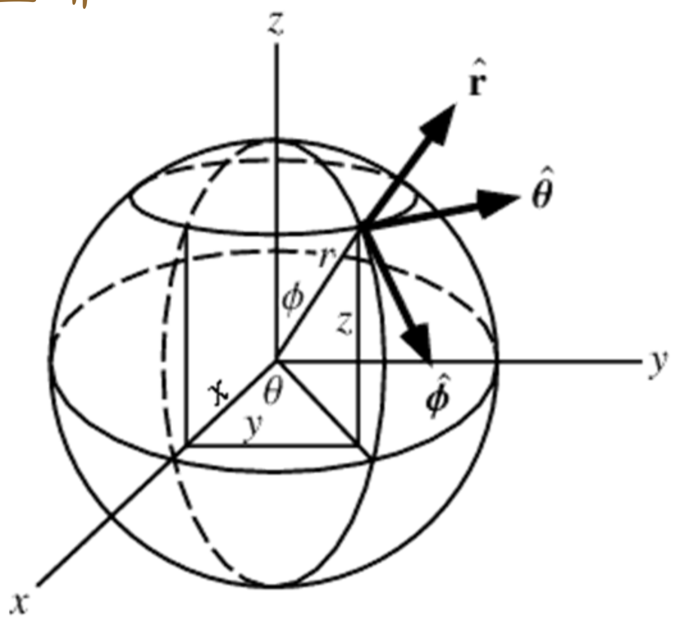
$$\rho = \sqrt{x^2 + y^2}$$

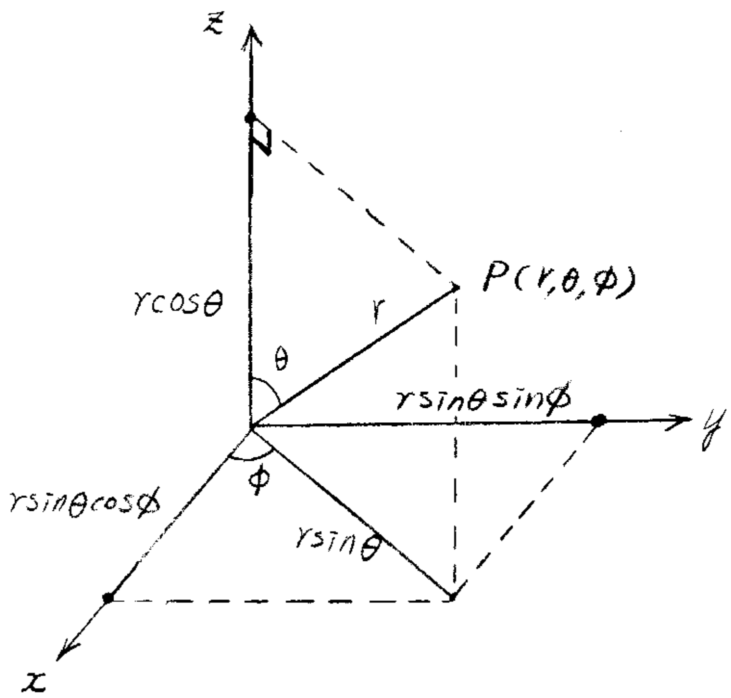
$$\phi = \tan^{-1}\frac{y}{x} = \cos^{-1}\frac{x}{\rho} = \sin^{-1}\frac{y}{\rho}$$

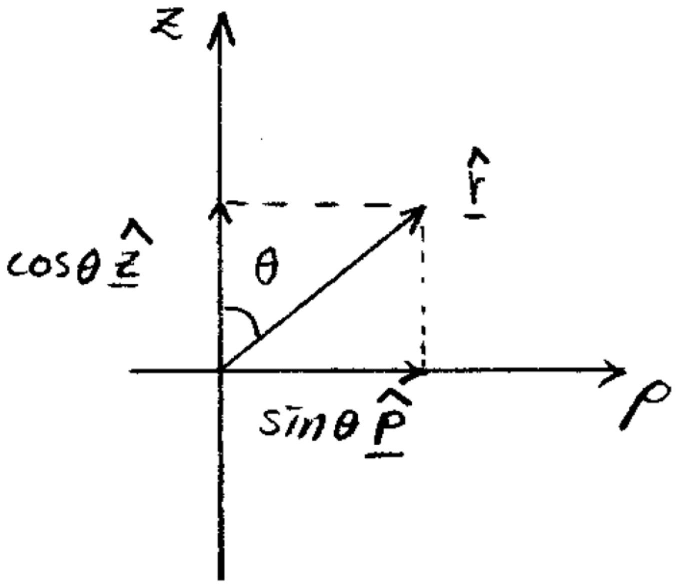


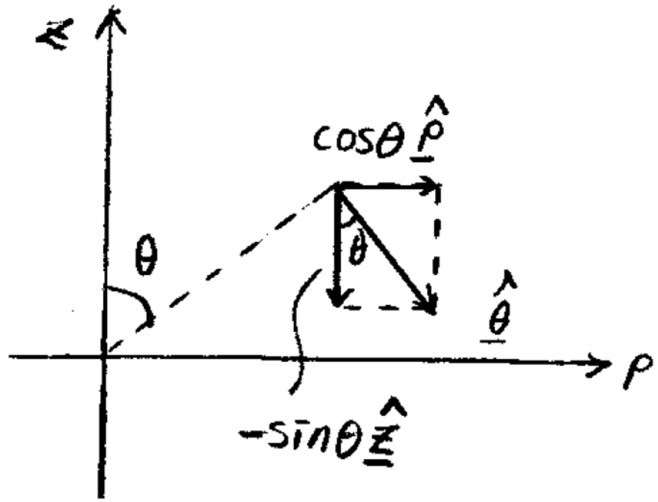


- 구좌표계


z
r̂
θ̂
r
ϕ
z
y
x
θ
y
ϕ̂
x


z
P(r,θ,ϕ)
rcosθ
r
θ
rsinθsinϕ
y
ϕ
rsinθcosϕ
rsinθ
x

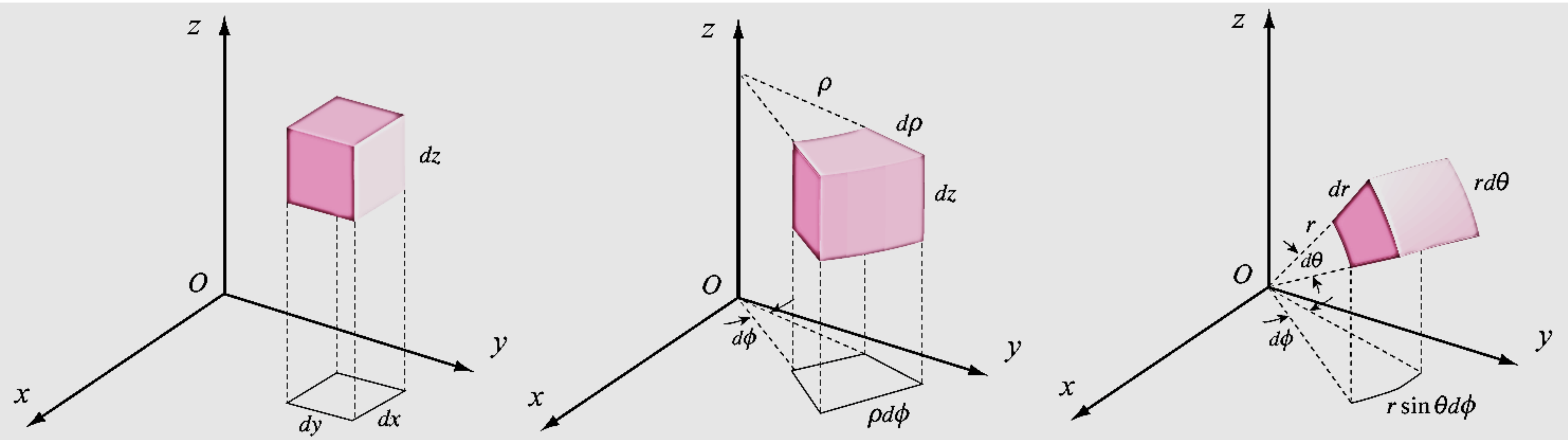

z
r̂
cosθ ẑ
θ
sinθ ρ̂
ρ


z
cosθ ρ̂
θ
θ
θ̂
ρ
-sinθ ẑ

7. Infinitesimal Surface and Length

- 각 좌표계에서 극소체적의 식

적분을 위해서 매우 작은 체적을 수식으로 표현할 필요가 있음



$$dv = dx\,dy\,dz\ (\mathrm{m}^3)$$

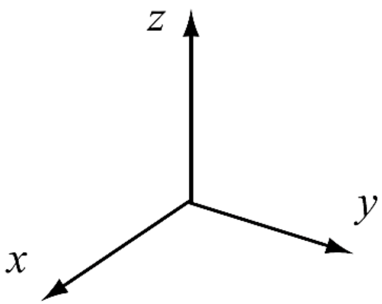
$$dv = d\rho(\rho d\phi)dz = \rho d\rho d\phi dz\ (\mathrm{m}^3)$$

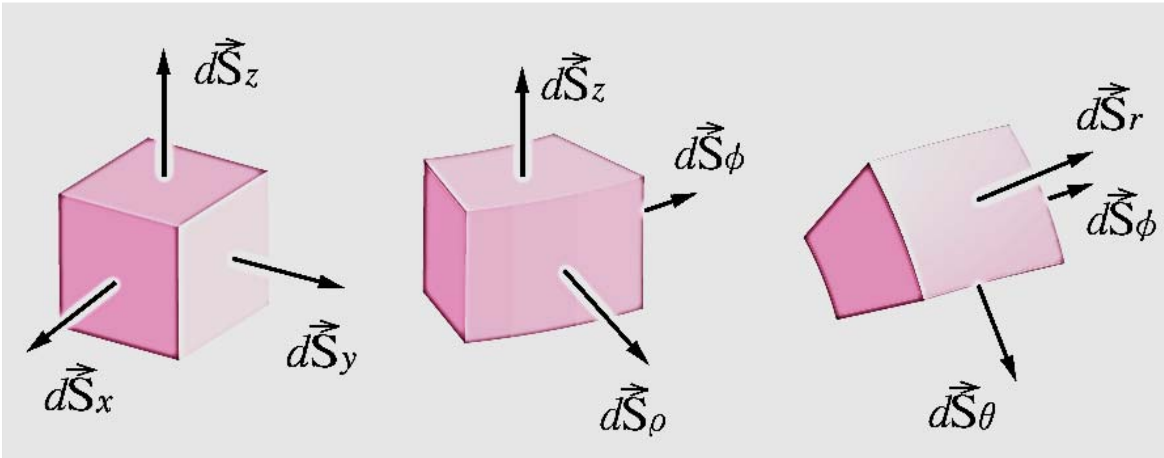
$$dv = dr(r d\theta)(r\sin\theta d\phi) = r^2\sin\theta\, dr\, d\theta\, d\phi\ (\mathrm{m}^3)$$

직교좌표계　　　원통좌표계　　　구좌표계

▪ 극소면적의 식

적분을 위해서 매우 작은 면적을 수식으로 표현할 필요가 있음





$$d\vec{S}_x = dy\,dz\,\hat{\mathbf{x}} \qquad d\vec{S}_\rho = \rho\,d\phi\,dz\,\hat{\boldsymbol{\rho}} \qquad d\vec{S}_r = r^2 \sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$$

$$d\vec{S}_y = dx\,dz\,\hat{\mathbf{y}} \qquad d\vec{S}_\phi = d\rho\,dz\,\hat{\boldsymbol{\phi}} \qquad d\vec{S}_\theta = r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$$

$$d\vec{S}_z = dx\,dy\,\hat{\mathbf{z}} \qquad d\vec{S}_z = \rho\,d\rho\,d\phi\,\hat{\mathbf{z}} \qquad d\vec{S}_\phi = r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$$

Example:

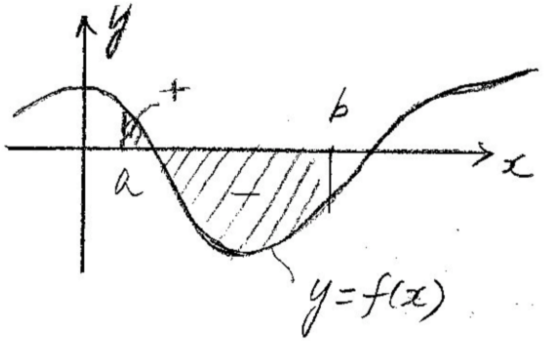
미소체적의 식을 이용해서 반지름이 a인 구의 체적을 구하라.

미소체적이 $dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$이므로 이를 적분하여,

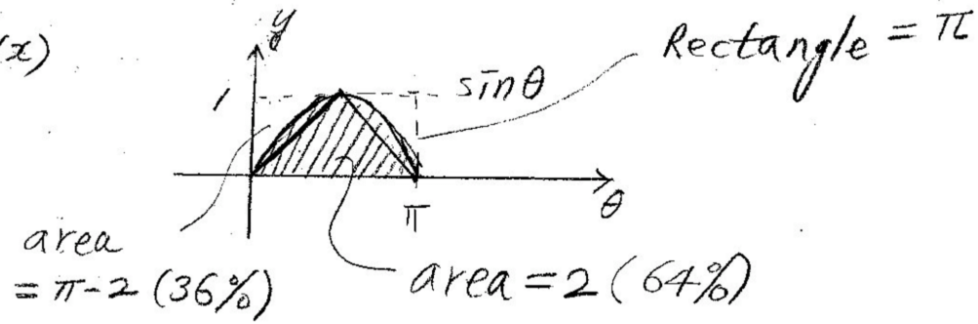
$$v = \int_v dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{4}{3}\pi a^3 \ \ (\mathrm{m}^3).$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta\,(dr\,d\theta\,d\phi) = \int_0^{a} r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \left[\frac{r^3}{3}\right]_0^a \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi} = \frac{a^3}{3}(2)(2\pi) = \frac{4\pi}{3}a^3$$

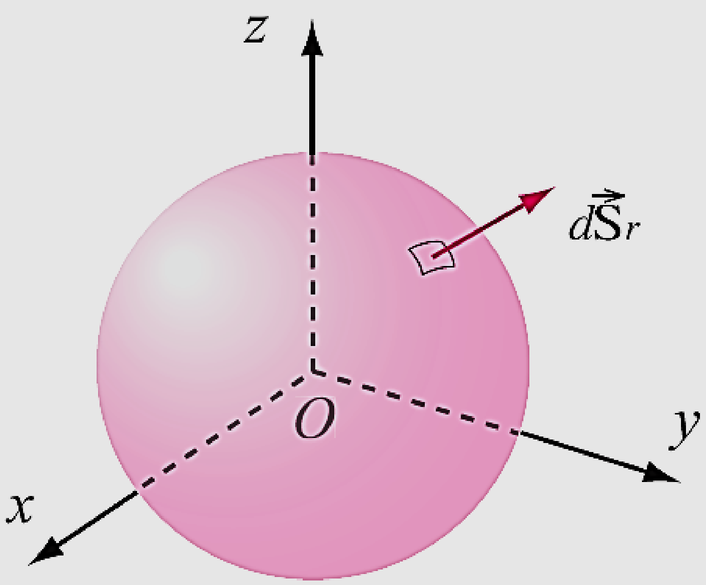


$\int_a^b f(x)\,dx$ = sum of singned areas



Example:

그림 E1-24의 지름이 r인 구의 표면적 S를 구하라.



$$\vec{S} = \int_S d\vec{S}_r = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}} = 4\pi r^2 \hat{\mathbf{r}} \ (\mathrm{m}^2)$$

$$S = \int_S dS_r = 4\pi r^2 \quad (\mathrm{m}^2)$$

8. Coding Example

Given the coordinate (x, y, z) in the rectangular coordinate of a point P,

1) find the coordinate (r, θ, φ) in the spherical coordinate of the point P and

2) express base vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ at P in terms of base vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$.

(Formulas)

1)

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \cos^{-1} \frac{z}{r}$$

$$\varphi = \tan^{-1} \frac{y}{x} = \cos^{-1} \frac{x}{\sqrt{x^2 + y^2}}$$

2)

$$\hat{\mathbf{r}} = \sin\theta \cos\phi \hat{\mathbf{x}} + \sin\theta \sin\phi \hat{\mathbf{y}} + \cos\theta \hat{\mathbf{z}}$$

$$\hat{\theta} = \cos\theta \cos\phi \hat{\mathbf{x}} + \cos\theta \sin\phi \hat{\mathbf{y}} - \sin\theta \hat{\mathbf{z}}$$

$$\hat{\phi} = -\sin\phi \hat{\mathbf{x}} + \cos\phi \hat{\mathbf{y}}$$

8. Coding Example

(Code)

```
# EM-Lec3-Coordinate Systems
# Given a point P(x,y,z),
# 1) express P in the spherical coordinate
# 2) express base vectors <r>,<theta>,<phi> of the spherical
# coordinate systems in terms of base vectors <x>,<y>,<z>

from math import *
rtd=180/3.141593
while True:
  x,y,z=map(float, input('x,y,z=').split())
  r=sqrt(x**2+y**2+z**2)

  theta=acos(z/r)
  rho=sqrt(x**2+y**2)
  phi=acos(x/rho)

  print('  (r,theta,phi)=(',r,',',theta*rtd,',',phi*rtd,')')

  rx=sin(theta)*cos(phi)
  ry=sin(theta)*sin(phi)
  rz=cos(theta)
```

```
  thetax=cos(theta)*cos(phi)
  thetay=cos(theta)*sin(phi)
  thetaz=-sin(theta)

  phix=-sin(phi)
  phiy=cos(phi)
  phiz=0

  print('  <r>    =(',rx,',',ry,',',rz,')')
  print('  <theta>=(',thetax,',',thetay,',',thetaz,')')
  print('  <phi>  =(',phix,',',phiy,',',phiz,')')
```

(코드실행결과)

```
x,y,z=
3 1 9
  (r,theta,phi)=( 9.539392014169456 , 19.359646526413474 , 18.43494679017798 )
  <r>    =( 0.3144854510165753 , 0.10482848367219182 , 0.9434563530497265 )
  <theta>=( 0.8950412845779244 , 0.29834709485930827 , -0.3314967720658978 )
  <phi>  =( -0.31622776601683805 , 0.9486832980505138 , 0 )
x,y,z=
```

Fin

(End)