Lecture 6 Electrostatics (3)

1. Free Electrons
2. Conductors and Electrostatic Field
3. Current and Resistance
4. Charge Continuity
5. Coding Example

1. Free Electrons

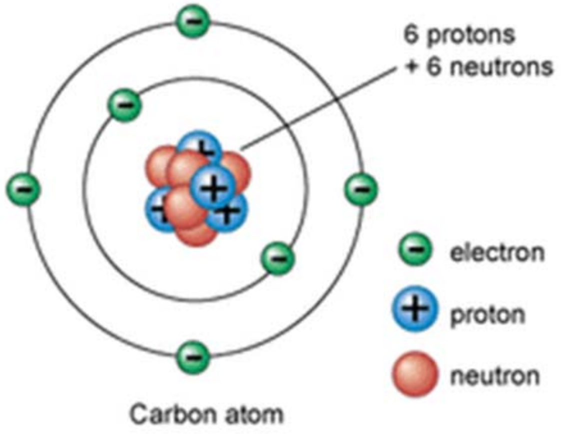
- Free electrons and conductors

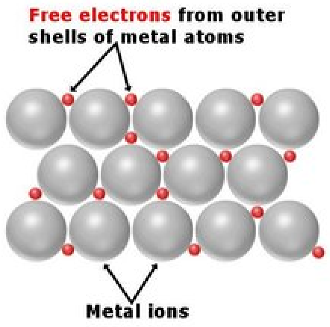
- Free electrons: fee to move by E field

- Conductors:

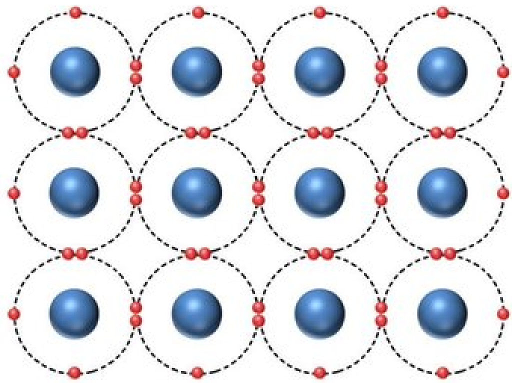
  Mostly metals: copper, aluminum, gold, silver

  Has a large free electron density: $n_e = 10^{29}$ /m^3

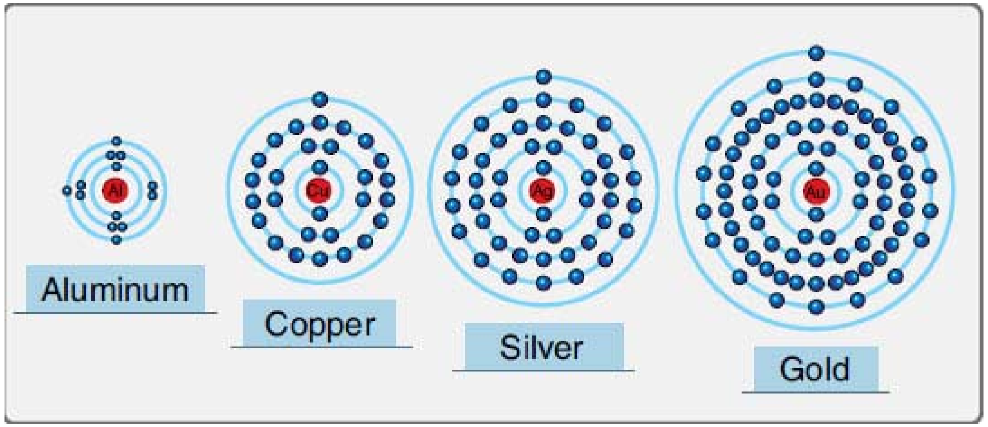




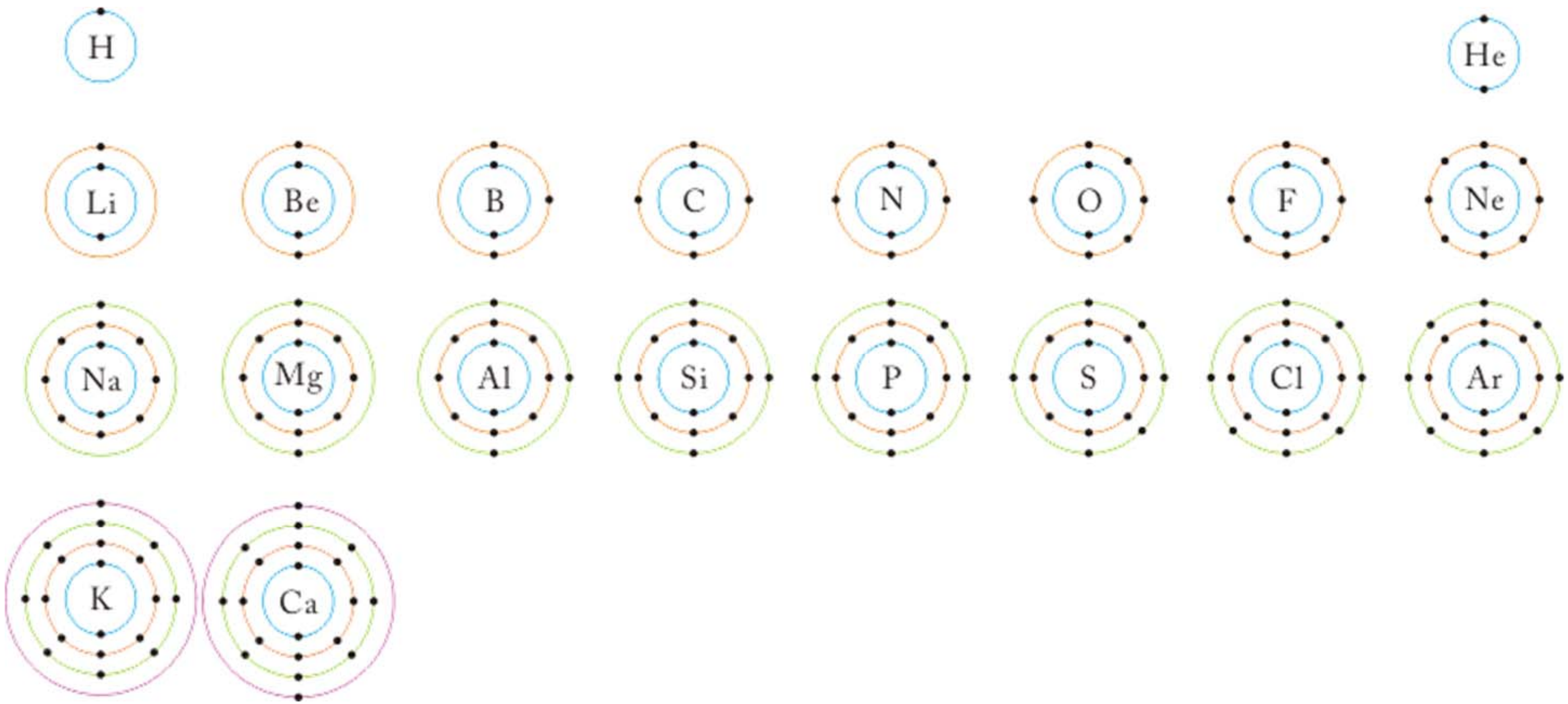
Conductor



Insulator



1. Free Electrons



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1. Free Electrons

- Example:

Cu density: 9×10^3 kg/m^3

Atomic mass: 63.5

Number of free electrons per atom: 1

Find the number n_e of free electrons per cubic meter.

$$\frac{\text{density}}{\text{molar mass}}=\frac{\text{Mass}}{\text{Atomic mass}}\times\frac{1}{\text{Volume}}=\frac{\text{Number of moles}}{\text{Volume}}$$

$$\frac{n}{v}=\frac{9\times10^3}{63.5}\text{kgm}^{-3}$$

$$n=1.4173\times10^5\,\text{moles}$$

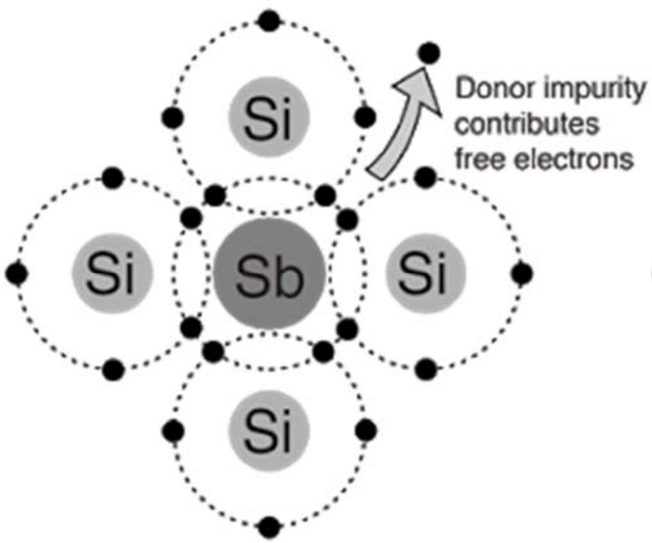
$$n_e=1.4173\times10^5\times6.023\times10^{23}=8.53\times10^{28}\,/\,\text{m}^3$$

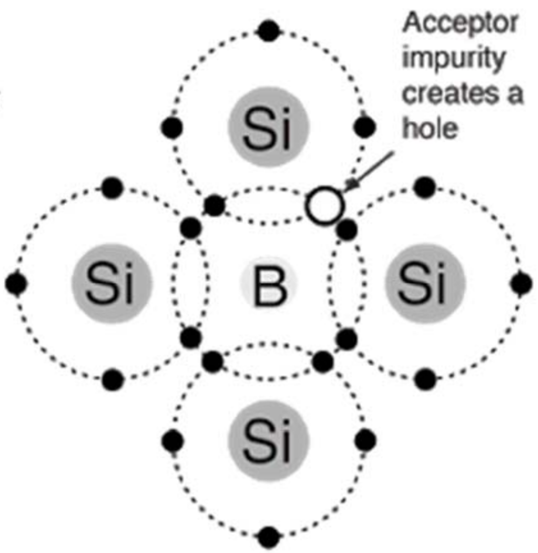
Semiconductors

- Silicon, gemanium
- Charge carrying mechanisms: electrons, holes, majority carrier, minority carrier
- *n*-type and *p*-type semiconductors
- Electron density smaller than conductors: n_e = **10^{10} (Si), 10^{13} (Ge) /m^3**
- Charge carrier density controlled by inserted impurities: 10^{16} - 10^{25} /m^3

$^{51}Sb_{121.760}$: antimony (안티모니) (합금, 반도체; 유독성)

$^{5}B_{10.811}$: boron (붕소) (내화유리, 살충제, 중성자흡수 제어봉, 고강성섬유, 고강도 화합물 기계 재료, 반도체)

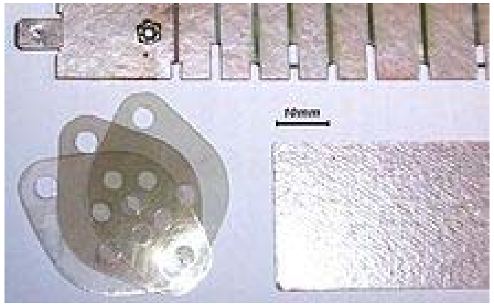




▪ Insulators

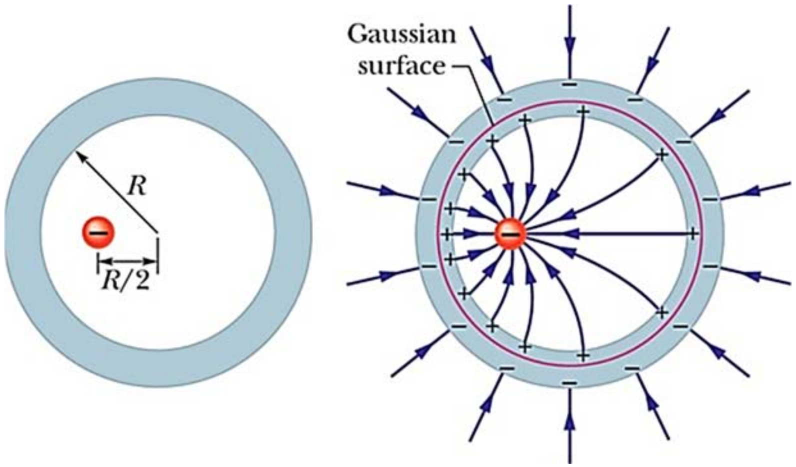
- Electron densiy much smaller than conductors: $n_e = 10^6\ /\mathrm{m}^3$
- Ceramic (porcelain), plastic
- Dielectric materials
- Dielectric breakdown: 20 (vacuum), 30 (air), 200 (PE), 100 (porcelain), 600 (Teflon), 1000 (glass), 2000 (mica) kV/cm
- Porcelain high-voltage insulator string

- Mica circuit insulator

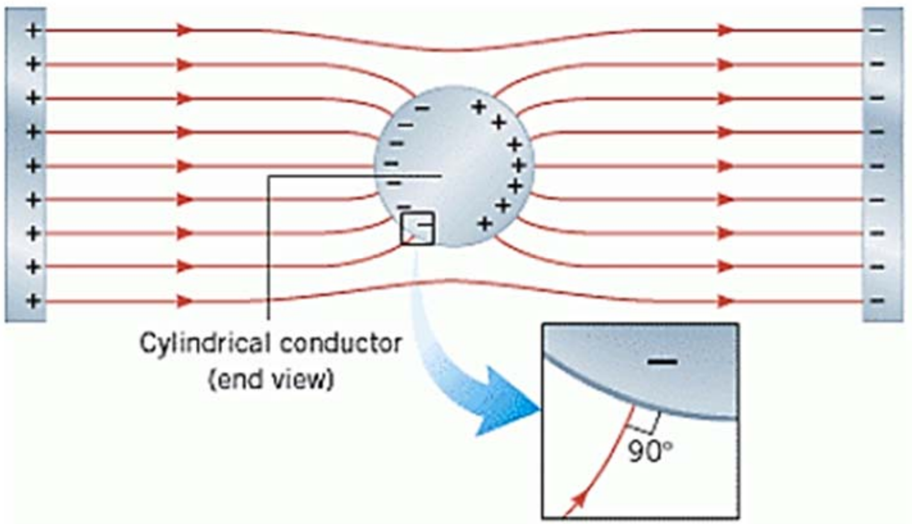


- Charge induction

- Induction by a field internal to a conductor



- Induction by a field external to a conductor



- Conductor and electric field

- PEC (perfect electric conductor)

- Infinite conductivity: $\boldsymbol{\sigma} = \infty$ (no work required to move a chage)

- Zero electric field inside: $\boldsymbol{E} = \mathbf{0}$ **(inside)**

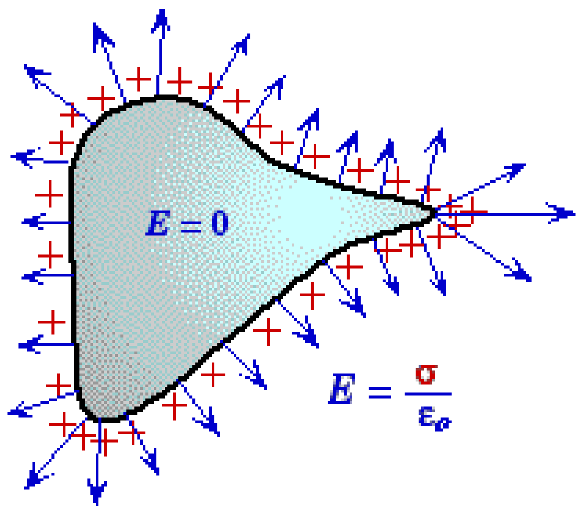
- Charge only on the surface if any: $\boldsymbol{\rho_s}$ = **surface charge density (C/m^2)**

- Zero charge density inside the conductor: $\boldsymbol{\rho_v} = \mathbf{0}$

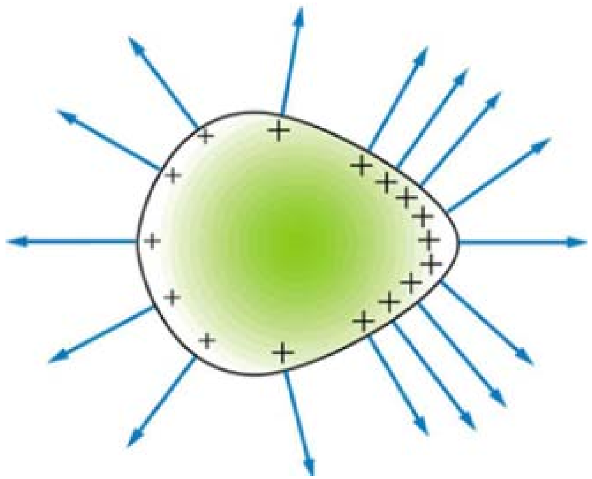
- Same potential inside the conductor and on the suface: V = **const**

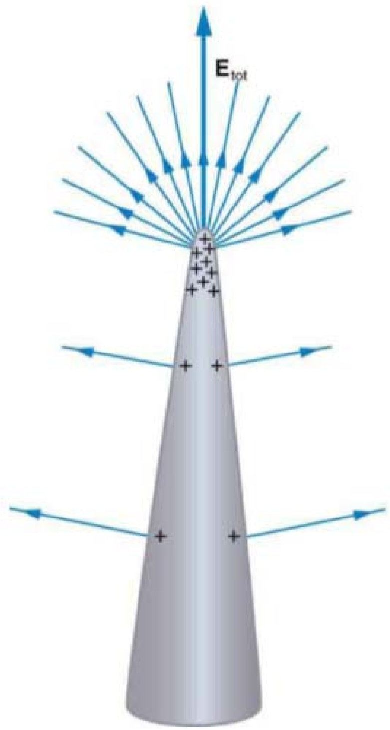
- Electric field normal to the surface: $\boldsymbol{E}_{\mathbf{tan}} = \mathbf{0}$

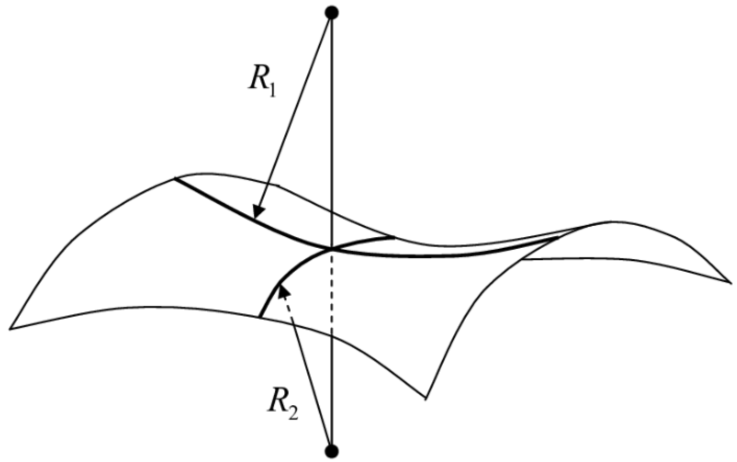
- Charge density related to flux density: $\boldsymbol{\rho_s} = \boldsymbol{D_n} = \boldsymbol{E_n} / \boldsymbol{\varepsilon}$



- Electric field on a conductor







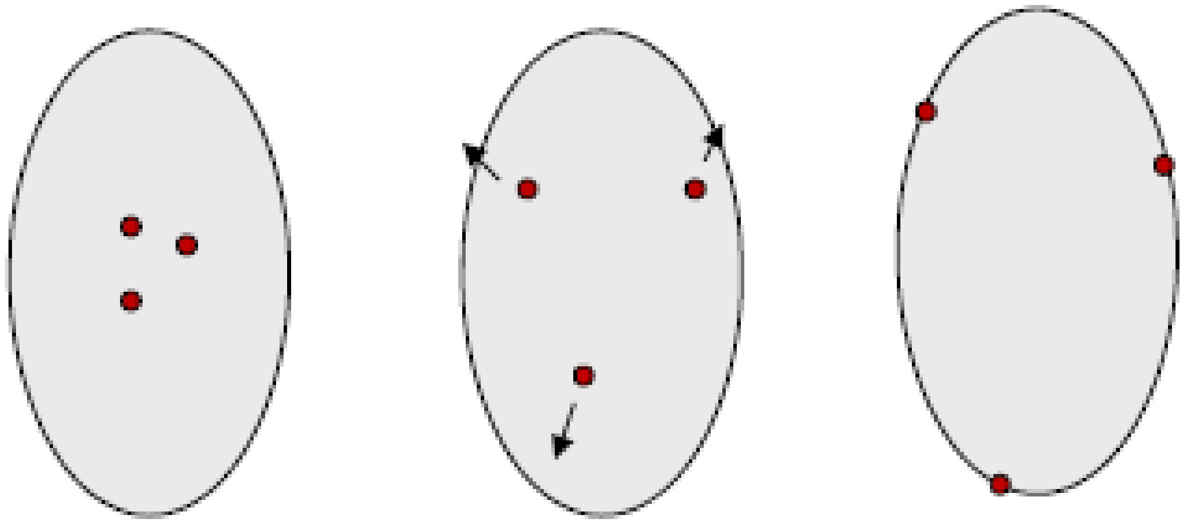
$$E = \frac{\rho_S}{\varepsilon}$$

$$\frac{dE}{dn} = -E\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$: Green's formula [1989, *Essay on Electricity and Magnetism*]

R_1, R_2 : principal radii of curvature of the surface

- Charge distribution on the conductor surface

1. Inject some charges of the same sign inside a conductor
2. Charges make E field
3. E field repels charges against each other
4. Charges move to the conductor surface until the only force on a charge is normal to surface. → Total E is normal to surface. $E_{tan} = 0 \rightarrow E_{tan} = \partial V/\partial t = 0 \rightarrow$ V = const on the conductor surface



- Summary of equations for conductors in electric field

$$\sigma = \infty$$

$$E_{\tan} = 0$$

$$E_n = \frac{D_n}{\varepsilon} = \rho_S$$

$$V = \text{const} \quad \text{on } S$$

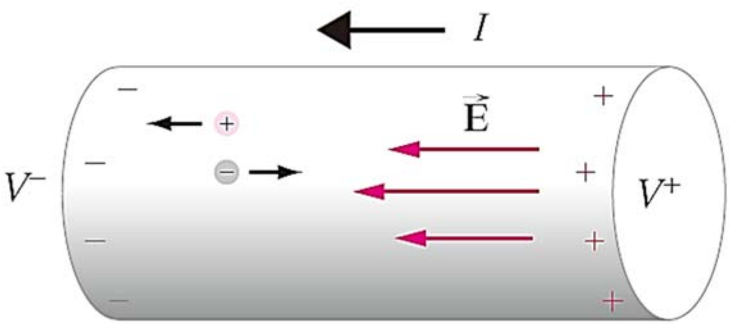
$$P = \frac{\varepsilon E^2}{2} = \frac{\rho_s^2}{2\varepsilon} \quad (\text{N/m}^2) : \text{pressure on the conductor}$$

- Current

- Potential difference in a material → **E** field

→ **F** = q**E** → charge movement → current (electric current) flow

- Definition of current



$$\boxed{i = \frac{dQ}{dt} \text{ (A)}}$$

Current, order of magnitude

10 uA : ventricular fibrillator (제세동기)

0.7 mA: hearid aid (보청기, 1.4 V 1 mW)

1 mA : threshold of human sensation

16 mA : maximum safe current of average man (grasp and letgo)

20 mA : LED current, human respiratory paralysis

100 mA : heart stops beating

200 mA : 220V 44W tungsten incandescent light bulb

10A : 220V electric kettle

100 A : motor vehicle starter

300 A : 30 V arc welder

400 A : KTX-산천, 10 MW, 25 kV 400A

2 kA : 115 kV electrical substation

30 kA : lightning strike

200 kA : electric arc furnace

10^{18} A: black-hole magnetic-field driven current

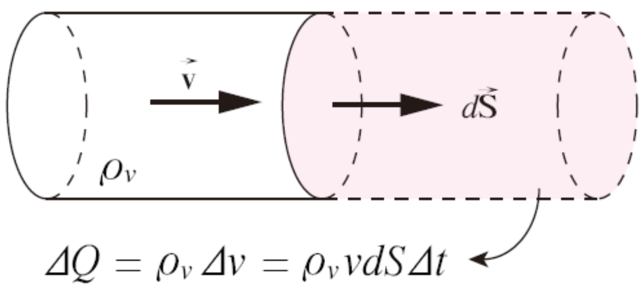
- Volume current density

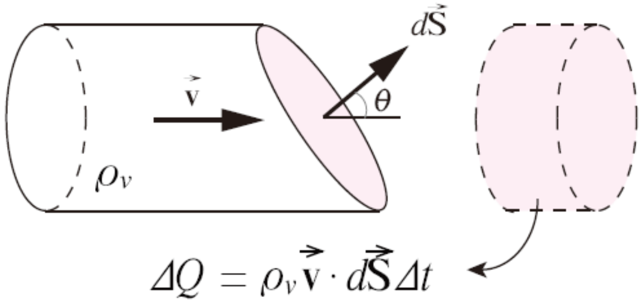
$$\Delta Q = \rho_v \Delta V = \rho_v (\Delta \ell)(dS) = \rho_v (v \Delta t)(dS)$$

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v v\, dS = \rho_v \mathbf{v} \cdot d\mathbf{S}$$

$\mathbf{J} = \rho_v \mathbf{v}$ (A/m^2) : volume current density

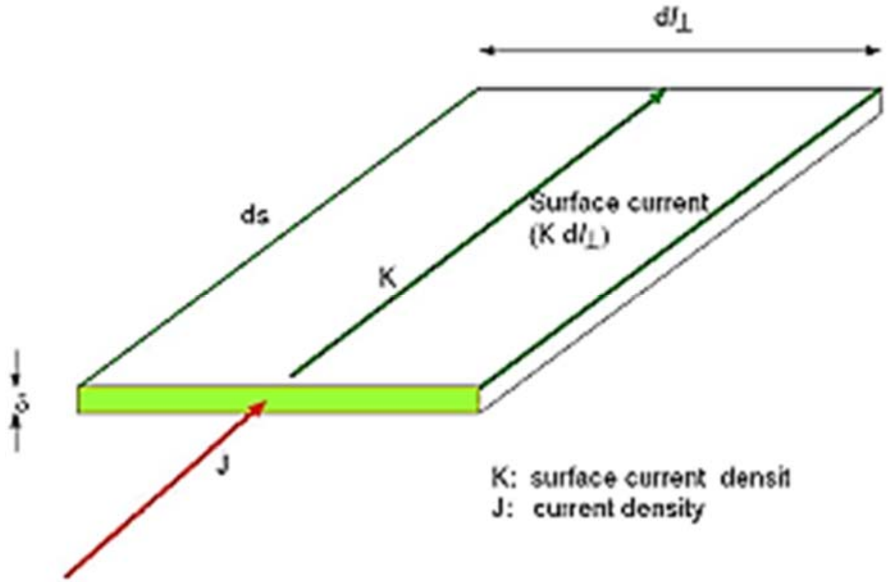
$$\boxed{\mathbf{J} = \frac{\Delta I}{\Delta S}}$$





- Surface current density

$\mathbf{J}_s = \frac{\Delta I}{\Delta W}$ (A/m)

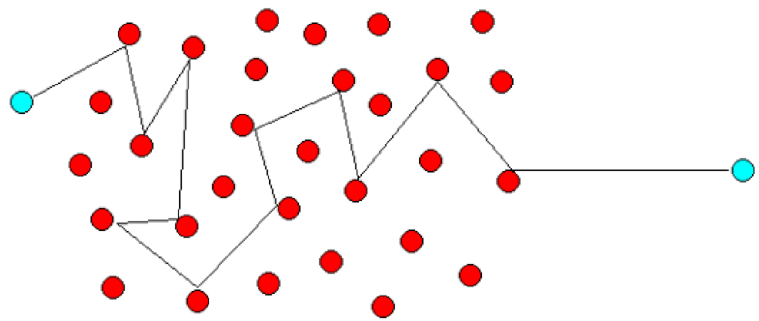


Electron drift velocity

- Fermi speed: random electron motion. v_F = 1.57 x 10^6 m/s (copper)
- Meam free path : average path length of an electron before collision.

  L_d = 4 x 10^{-8} m (Cu)
- Drift velocity: electron velocity under applied E field

$$\mathbf{v}_d = -\mu_e \mathbf{E} = -\frac{\mathbf{J}}{n_e e}$$



μ_e : electron mobility

$\mathbf{J}$: volume current density (A/m2)

n_e : free electron density (m^{-3})

e : charge of an electron. 1.6×10^{-19} C

- Conductivity and resistivity

$$\mathbf{J} = \rho_v \mathbf{v}_d = -\rho_v \mu_e \mathbf{E} \equiv \sigma \mathbf{E} \quad \text{(current density, A/m}^2\text{)}$$

$$\sigma = -\rho_v \mu_e \quad \text{(conductivity, S/m)}$$

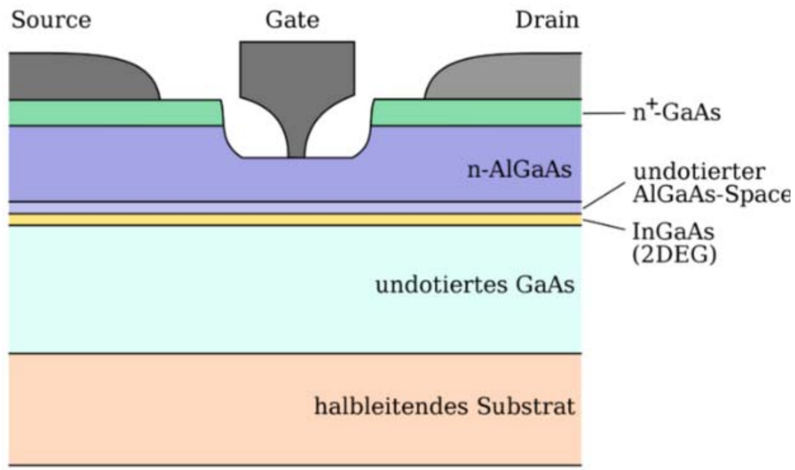
$$\rho_v = nq$$

$$\rho = \frac{1}{\sigma} \quad (\text{resistivity, ohm meter, } \Omega\,\text{m})$$

Electron mobility table

List of highest measured mobilities [cm^2/ (V·s)]

Material	Electron mobility	Hole mobility
AlGaAs/GaAs heterostructures	35,000,000[2]	
Freestanding Graphene	200,000[4]	
Carbon nanotubes	79,000[6][7]	
Crystalline silicon	1,400[1]	450[1]
Polycrystalline silicon	100	
Metals (Al, Au, Cu, Ag)	10-50	
2D Material (MoS2)	10-50	
Organics	8.6[8]	43[9]
Amorphous silicon	~1[10]	



- HEMT (high electron mobility transistor): used in low-noise, very high frequency amplifier

Electrial resistivity of materials

- Resistance wire: nichrome, 1 μohm-meter

  80/20 = 80% Ni + 20% Cr

  Menting point 1350°C

  Max. op. temp. 1150°C

  220V 2kW heater: wire temp. 300-500°C

  Resistivity about 60 times that of Cu

- Mica: 1×10^{13} ohm-meter
- Quartz: 10^{10} to 10^{17} ohm-meter
- Diamond: 10^{11} to 10^{18} ohm-meter
- Water:

  Pure: 2×10^{9}

  Distilled: 5×10^{6}

  River water: 20,000

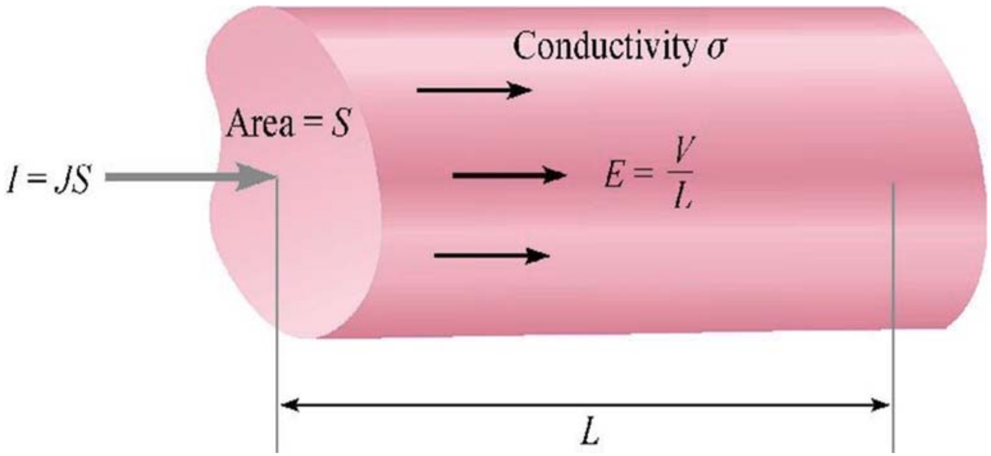
  Sea water: 2,000

Material	Resistivity ρ (ohm m)
Silver	1.59×10^{-8}
Copper	1.68×10^{-8}
Copper, annealed	1.72×10^{-8}
Aluminum	2.65×10^{-8}
Tungsten	5.6×10^{-8}
Iron	9.71×10^{-8}
Platinum	10.6×10^{-8}
Manganin	48.2×10^{-8}
Lead	22×10^{-8}
Mercury	98×10^{-8}
Nichrome (Ni.Fe.Cr alloy)	100×10^{-8}
Constantan	49×10^{-8}
Carbon* (graphite)	$3\text{-}60 \times 10^{-5}$
Germanium*	$1\text{-}500 \times 10^{-3}$
Silicon*	0.1-60 ...
Glass	$1\text{-}10000 \times 10^{9}$
Quartz (fused)	7.5×10^{17}
Hard rubber	$1\text{-}100 \times 10^{13}$

- Resistance and Ohm's law

- Cylindrical material with constant cross section

$$R \equiv \frac{V}{I} = \frac{-\int_b^a \mathbf{E} \cdot d\mathbf{L}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}} = \frac{EL}{E\sigma S} = \frac{L}{\sigma S} \quad \text{(resistance, ohm, } \Omega\text{)}$$



- Example

Copper wire of 1-mm diameter and 1-m length. 1 V applied between both ends

Resistance = 0.0216 ohm

Current density = $5.9 \times 10^7 A/m^2$

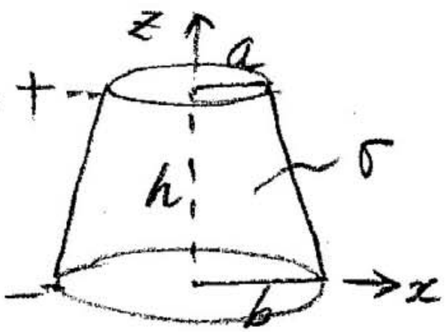
Electron drift velocity = 4.3 mm/s

- Resistance of materials with non-uniform cross section

$$dR = \frac{dL}{\sigma(L)S(L)}$$

$$R = \int_L \frac{dL}{\sigma(L)S(L)}$$

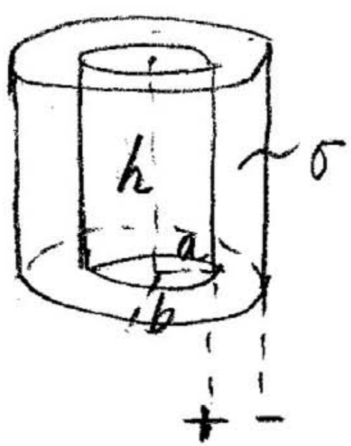
- Examples: resistance calculation

1. 

$$S = \pi r^2 = \pi\left[\frac{z}{h}(a-b)+b\right]^2$$

$$R = \frac{1}{\sigma}\int_0^h \frac{dL}{S(L)} = \frac{1}{\pi\sigma}\int_0^h \frac{dz}{\left[\frac{z}{h}(a-b)+b\right]^2} = \frac{1}{\pi\sigma}\frac{-h}{a-b}\left[\frac{1}{\frac{z}{h}(a-b)+b}\right]_0^h$$

$$= \frac{1}{\pi\sigma}\frac{h}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{\sigma}\frac{h}{\pi ab}$$

2. 

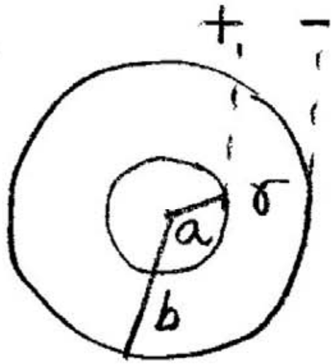
$$S = (2\pi\rho)h,\ dL = d\rho$$

$$R = \frac{1}{\sigma}\int_a^b \frac{d\rho}{S(\rho)} = \frac{1}{\sigma}\frac{1}{2\pi h}\int_a^b \frac{1}{\rho}d\rho = \frac{1}{\sigma}\frac{1}{2\pi h}\left[\ln\rho\right]_a^b$$

$$= \frac{1}{\sigma}\frac{\ln(b/a)}{2\pi h}$$

- Examples: resistance calculation

3.



$$S = 4\pi r^2,\; dL = dr$$

$$R = \frac{1}{\sigma}\int_a^b \frac{dr}{S(r)} = \frac{1}{\sigma}\frac{1}{4\pi}\int_a^b \frac{1}{r^2}dr = \frac{1}{\sigma}\frac{1}{4\pi}\left[-\frac{1}{r}\right]_a^b$$

$$= \frac{1}{\sigma}\frac{1}{4\pi}\left(\frac{1}{a}-\frac{1}{b}\right)$$

- Caution in Unit Change

- Example 1:

$\rho = 10\ \Omega\ \mathrm{m} = 10\ \Omega\ 100\ \mathrm{cm} = 1000\ \Omega\ \mathrm{cm}$. Wrong!

$\rho = 1\ \Omega\ \mathrm{cm} = 1\ \Omega\ 0.01\ \mathrm{m} = 0.01\ \Omega\ \mathrm{m}$. Correct!

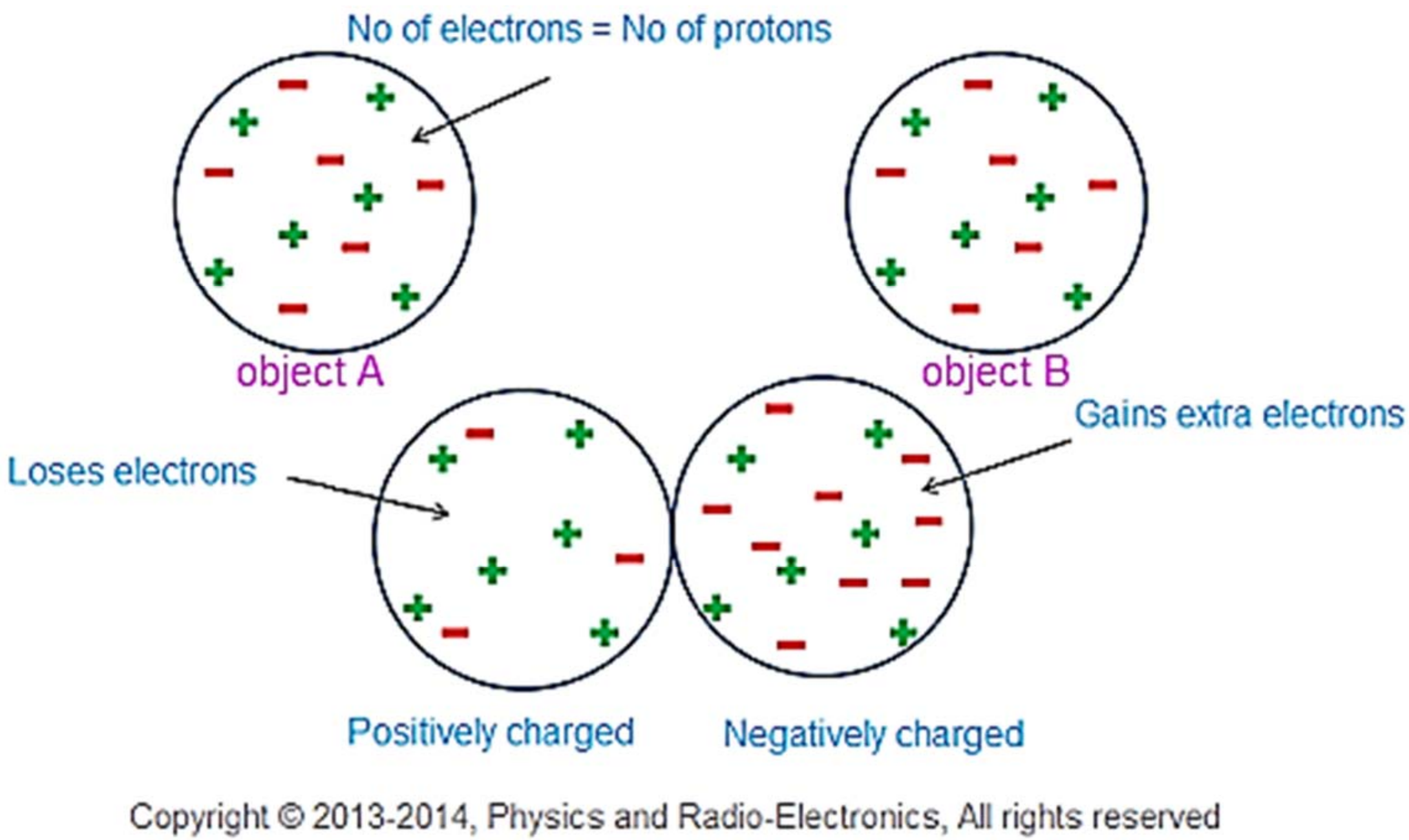
- Examplex 2:

$\sigma = 1\ \mu\mathrm{S/cm} = 10^{-6}\ \mathrm{S/(0.01m)} = 10^{8}\ \mathrm{S/m}$

4. Charge Continuity

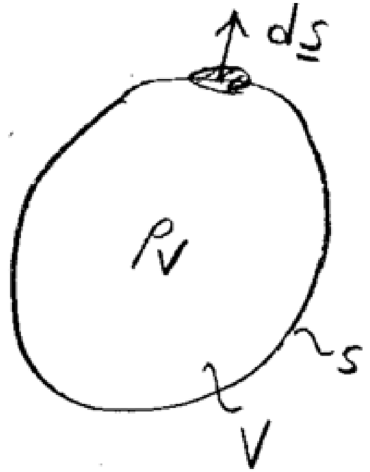
- Charge conservation law

- The net charge of an isolated system will remain constant.



- Charge continuity

- The total current flowing out of a closed surface is the negative of rate of change of charge inside the surface



$$I = -\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial t}\int_V \rho_v dV$$

$$I = \int_S \mathbf{J}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{J}dV = -\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial t}\int_V \rho_v dV$$

$$\boxed{\nabla\cdot\mathbf{J} = -\frac{\partial \rho_v}{\partial t}}$$

Charge relaxation

- Charge density decays exponentially in a conductor.

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} = \nabla \cdot \sigma \mathbf{E} = \nabla \cdot \frac{\sigma}{\varepsilon} \mathbf{D} = \frac{\sigma}{\varepsilon} \nabla \cdot \mathbf{D} = \frac{\sigma}{\varepsilon} \rho_v$$

$$\rho_v + \frac{\varepsilon}{\sigma} \frac{d\rho_v}{dt} = 0$$

$$\boxed{\rho_v = \rho_0 e^{-t/(\varepsilon/\sigma)} = \rho_0 e^{-t/\tau}}$$

$$\boxed{\tau = \frac{\varepsilon}{\sigma}} \text{ (relaxation time)}, \varepsilon = \varepsilon_0 \text{ in metals}$$

$$\tau = 1.5 \times 10^{-19} \text{s (copper)}, \tau = 252 \text{ s (porcelain)}$$

5. Coding Example

- Example 1:

Given a line charge at (0, 0, z_s) m with charge density $4\pi\varepsilon_0$ (C/m) with $a < z_s < b$ and a field point $P(x, 0, z)$ m,

1) find the potential V at P by numerical integration

2) find the potential V at P by formula

(Answer)

By numerical integration:

$$V = \frac{1}{4\pi\varepsilon}\int_a^b \frac{\rho_L}{R}dz = \frac{\rho_L}{4\pi\varepsilon}\Delta\sum_{k=0}^{n-1}\frac{1}{R_k}$$

$$R_k = \sqrt{(z - z_k) + x^2},\ z_k = a + (k + 0.5)\Delta$$

By formula:

$$V = \frac{\rho_L}{4\pi\varepsilon}\log\frac{R_b + b - z}{R_a + a - z}$$

$$R_b = \sqrt{(z - b)^2 + x^2},\ R_a = \sqrt{(z - a)^2 + x^2}$$

5. Coding Example

```
# EM-Lec6-Example 1: Electric potential due to a finite line charge
# Given a line charge at (0,0,zs) m with charge density 4pi*epsi0 (C/m)
# with a<zs<b and a field point P(x,0,z) m,
# 1) find the potential V at p by numerical integration
# 2) find the potential V at p by formula
from math import *
while True:
  a,b=map(float,input('Line charge at (0,0,zs) with zs from a to b: a, b (m) =').split())
  x,z=map(float,input('Field point at (x,0,z): x, z (m) =').split())
  n=int(input('Number of sampling points for numerical integration: n='))
  dzs=(b-a)/n
  zs=a-0.5*dzs
  sum=0
  for i in range(n):
    zs=zs+dzs
    r=sqrt((x-0)**2+(0-0)**2+(z-zs)**2)
    sum=sum+1/r
  sum=sum*dzs
  print('  Numerical integration: V(x,z)=',sum)
  rb= sqrt((x-0)**2+(0-0)**2+(z-b)**2)
  ra=sqrt((x-0)**2+(0-0)**2+(z-a)**2)
  v=log((rb+b-z)/(ra+a-z))
  print('  Formula: V(x,z)=',v)
'''
Line charge at (0,0,zs) with zs from a to b: a, b (m) =
-1 2
Field point: x, z (m) =
3 4
Number of sampling points for numerical integration: n=
10
  Numerical integration: V(x,z)= 0.658584955797933
  Formula: V(x,z)= 0.6586505454927757
Line charge at (0,0,zs) with zs from a to b: a, b (m) =
'''
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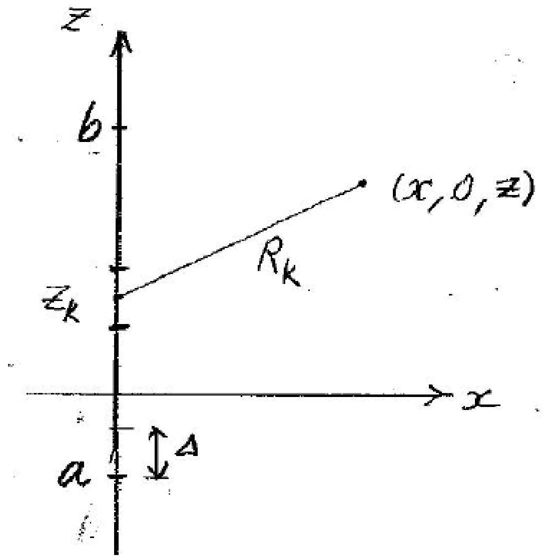

Fin

(End)