Lecture 7 Electrostatics (4)

1. Dielectric Polarization
2. Method of Images
3. Electrostatic Energy
4. Boundary Conditions
5. Coding Example

- Dielectric polarization

- Negative/positive charges oriented toward the positive/negative electrode
- Charges do not move freely in the material but reoritented.

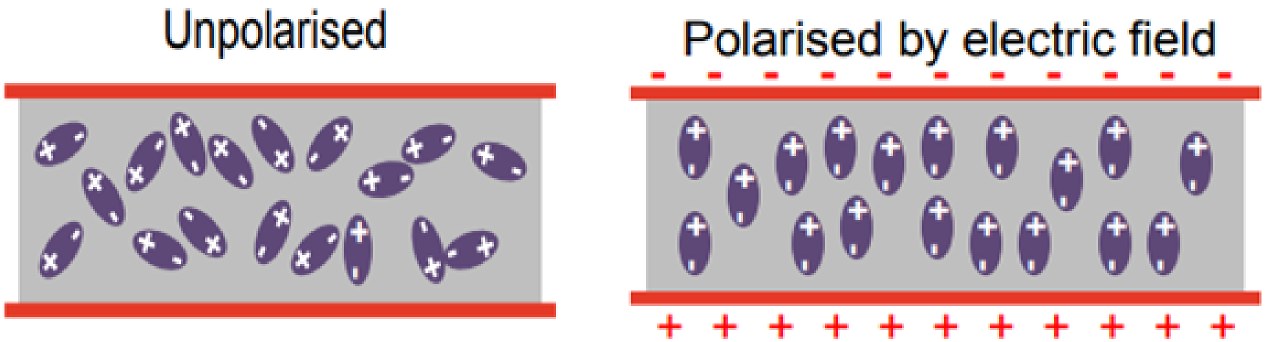
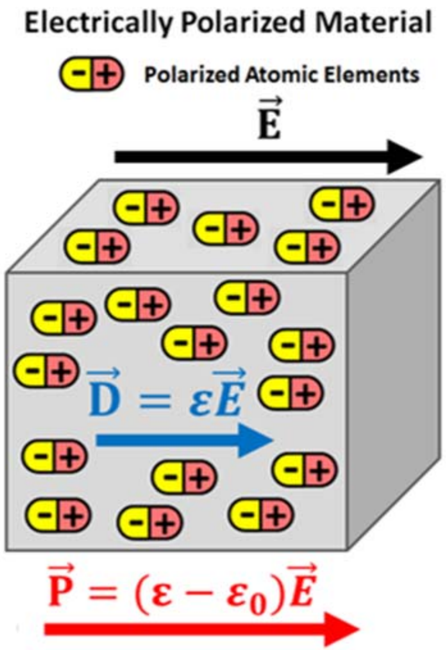


Fig. Polarization in materials [www.pcbdirectory.com]

- Permittivity of dielectric materials

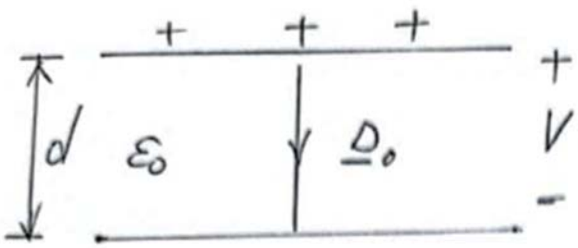


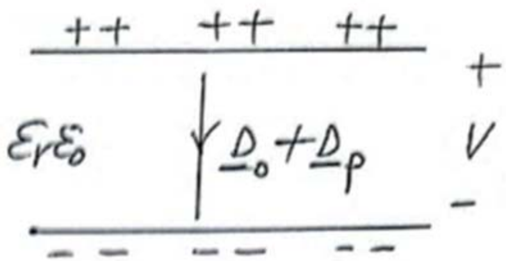
$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon_r\varepsilon_0\mathbf{E} = \varepsilon\mathbf{E}$$

ε_0 : permittivity of vacuum, 8.854×10^{-19} F/m

$\varepsilon = \varepsilon_r\varepsilon_0$: permittivity of dielectric materials

$\varepsilon_r = \varepsilon/\varepsilon_0$: dielectric constant

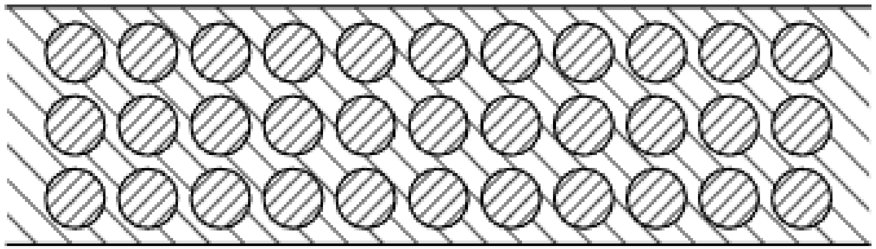




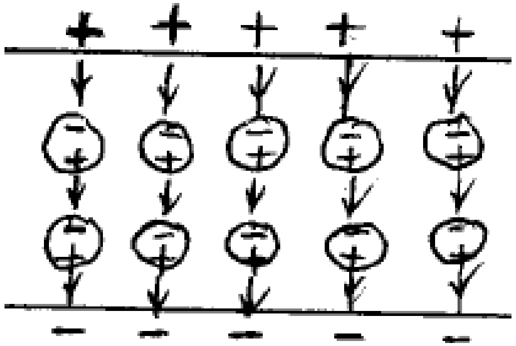
$$E = \frac{V}{d} \rightarrow D_0 = \varepsilon_0 E = \varepsilon_0\frac{V}{d} \qquad E = \frac{V}{d} \rightarrow D = D_0 + D_p = \varepsilon_r\varepsilon_0 E = \varepsilon_r\varepsilon_0\frac{V}{d} > D_0$$

▪ Dielectric polarization - Feynman's model

- A model of a dielectric: small conducting spheres embedded in an idealized insulator



- Charging of conducting spheres: top = negatively charged, bottom = positively charged



- Polarization (= charge separation) increases electric flux density in the material.

Polarization mechanisms in dielectric materials

- *Electron* displacements
- *Ion* displacements
- *Dipole* orientation
- *Space (interfacial) charge* displacements

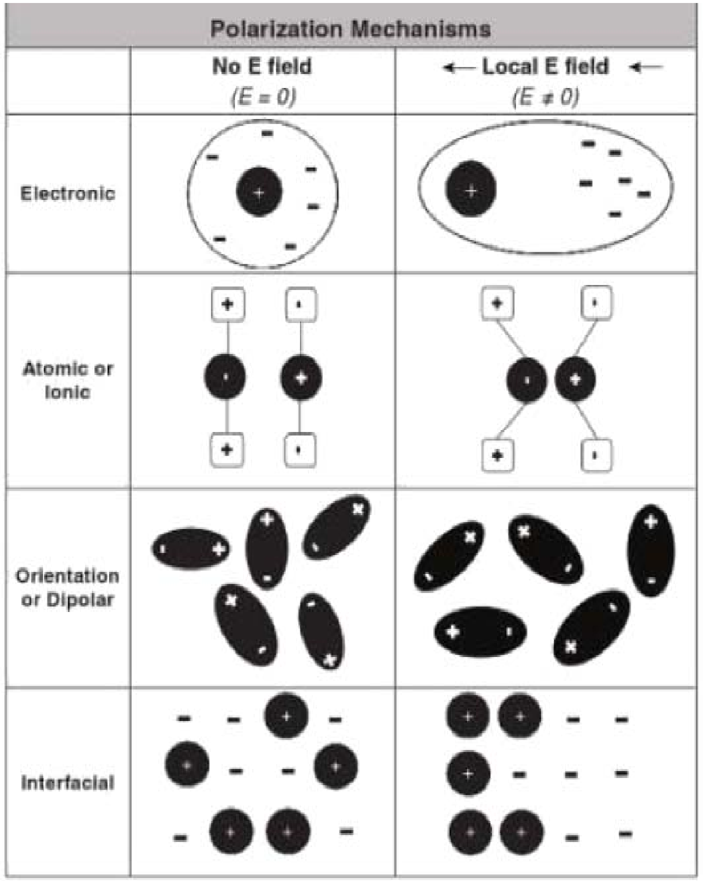
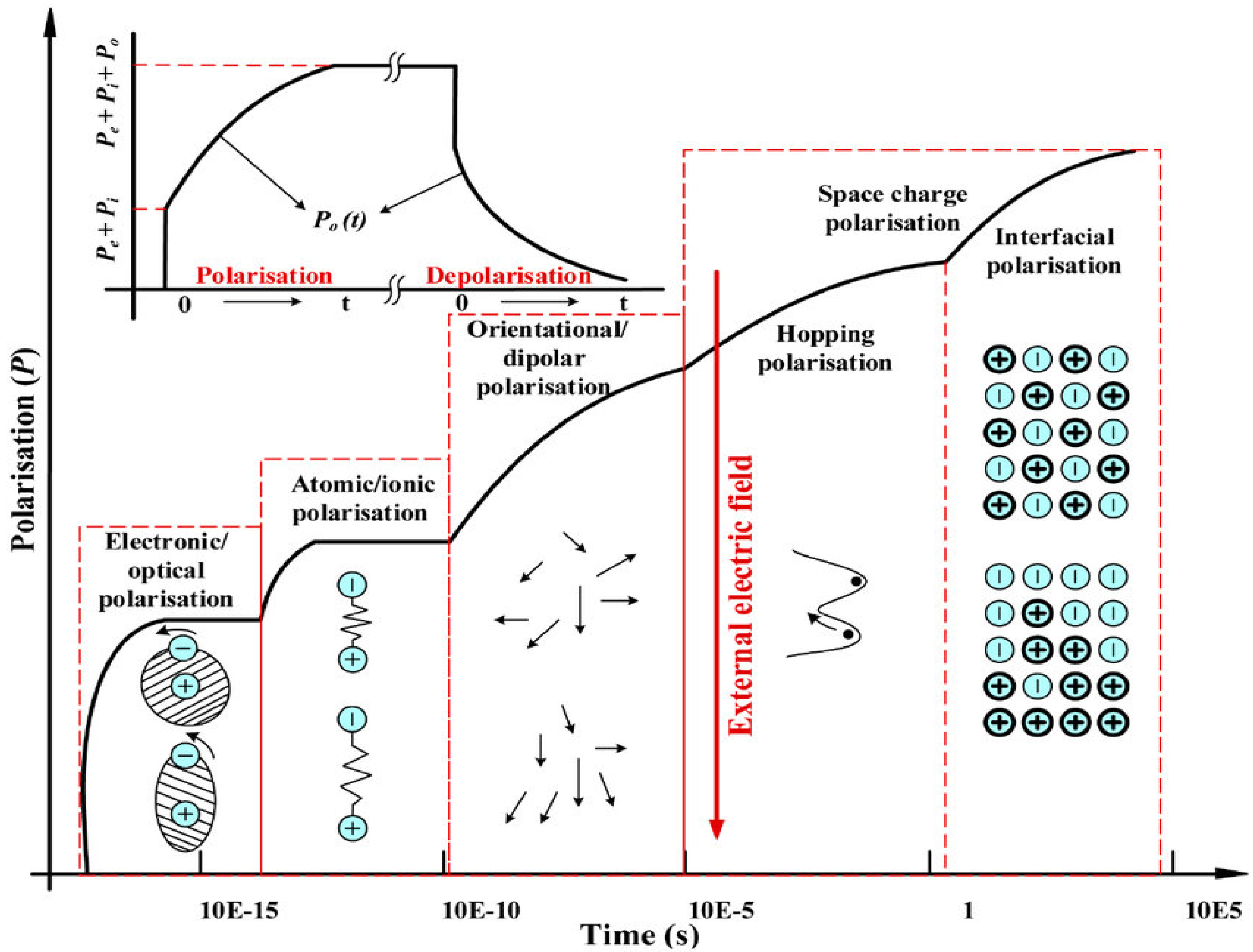


Fig. Polarization mechanisms [Knowles]

▪ Time constant of electric polarization

- Electric polarization: due to many different processes
- Polarization process: has a time constant (relaxation time)



▪ Polarization charge density

- The effect of polarization is modelled using charge densities.
- *Polarization charges* cannot move around. → *Bound charges*

$$\mathbf{P} = \text{ polarization} \equiv \text{ dipole moment per unit volume}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{R^2} \text{ (single dipole)}$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\mathbf{P}\cdot\hat{\mathbf{R}}}{R^2}dV' \text{ (distributed dipoles)}$$

$$V = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\mathbf{P}\cdot d\mathbf{S}'}{R} + \frac{1}{4\pi\varepsilon_0}\int_V \frac{(-\nabla'\cdot\mathbf{P})}{R}dV'$$

$$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} \text{ (polarized bound surface charge density)}$$

$$\rho_b \equiv -\nabla'\cdot\mathbf{P} \text{ (polarized bound volume charge density)}$$

- Electric flux density in dielectric materials

$$\rho = \rho_b + \rho_f$$

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f \ \text{(Gauss's law)}$$

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

$$\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P} \ \text{(electric flux density)}$$

$$\nabla \cdot \mathbf{D} = \rho_f$$

Permittivity of dielectric materials

$$\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}$$

$$\varepsilon \equiv \varepsilon_0 (1 + \chi_e) \text{ (permittivity)}$$

$$\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0} \text{ (dielectric constant or relative permittivity)}$$

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H_2)	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N_2)	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	$KTaNbO_3$ (0° C)	34,000

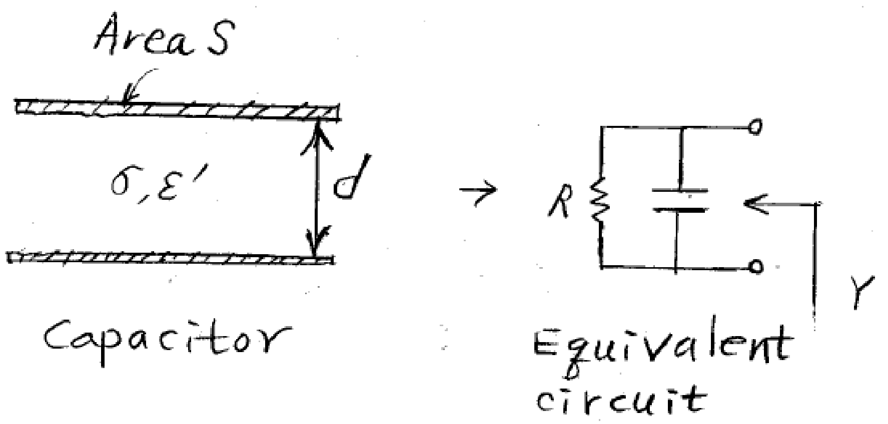
- Dielectric loss

$$Y = G + j\omega C = \frac{\sigma S}{d} + j\omega\varepsilon\frac{S}{d} = j\omega\frac{S}{d}(\varepsilon' - j\varepsilon'') \rightarrow \varepsilon'' = \frac{\sigma}{\omega}$$

$$\omega = 2\pi f$$

$$G = \frac{\sigma S}{d},\ C = \varepsilon'\frac{S}{d}$$

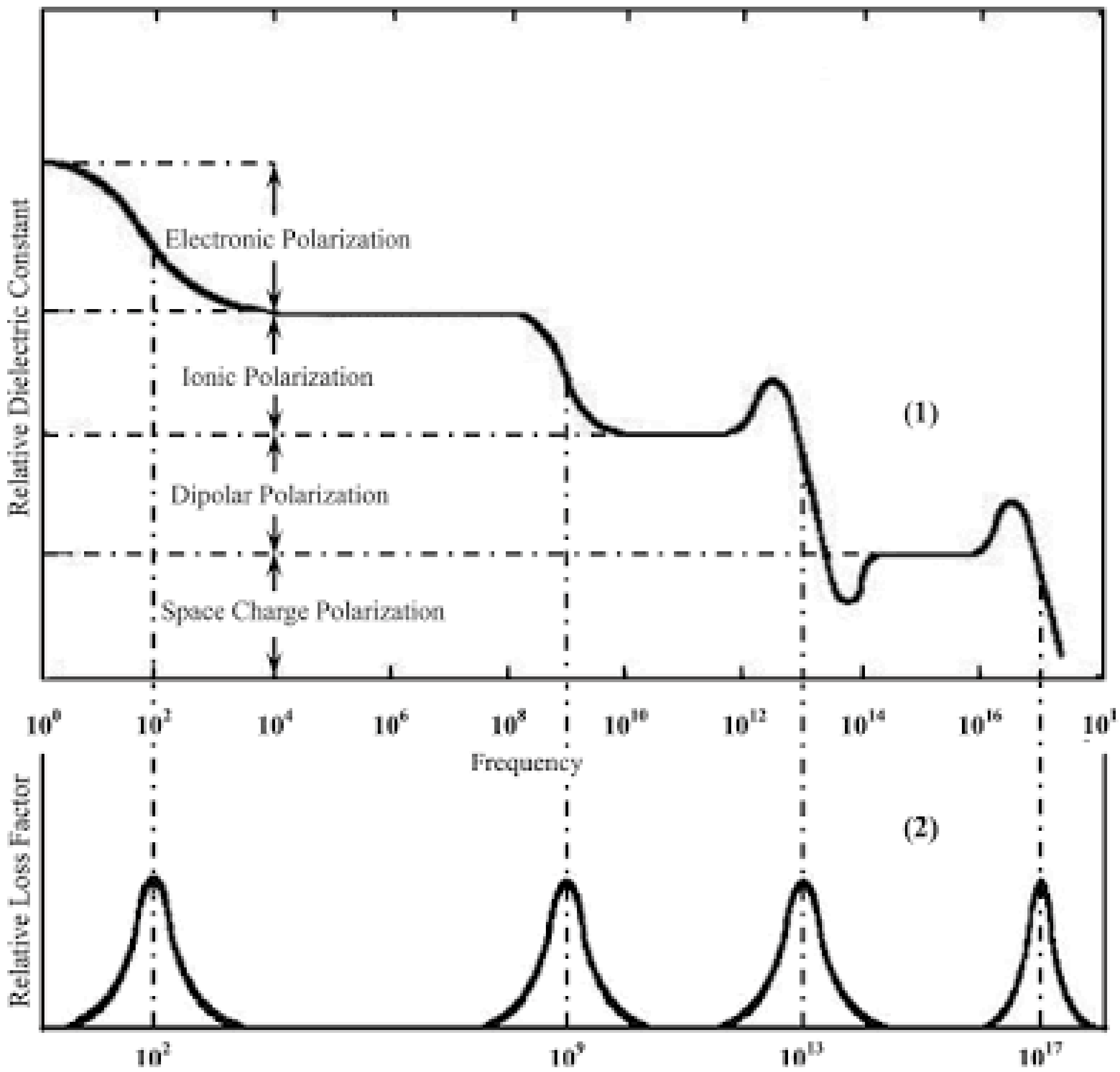
$$Q = CV,\ i = \frac{dQ}{dt} = C\frac{dV}{dt} \rightarrow I = j\omega CV \rightarrow Y = \frac{I}{V} = j\omega C$$



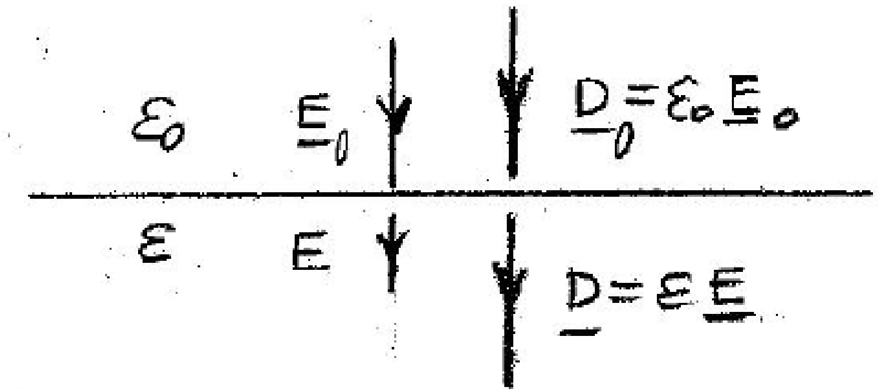
$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_0(\varepsilon_r' - j\varepsilon_r'')$$ (complex permittivitty)

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'}$$ (loss tangent)

- Permittivity dispersion: frequency response



- Example: dielectric in an external field



$$\mathbf{D} = \mathbf{D}_0 \quad \text{(continuity of electric flux; Gauss's law)}$$

$$\mathbf{E} = \frac{\varepsilon_0}{\varepsilon}\mathbf{E}_0$$

$$|\mathbf{E}| < |\mathbf{E}_0|$$

2. Method of Images

$\mathbf{r} = (x, y, z)$: field point

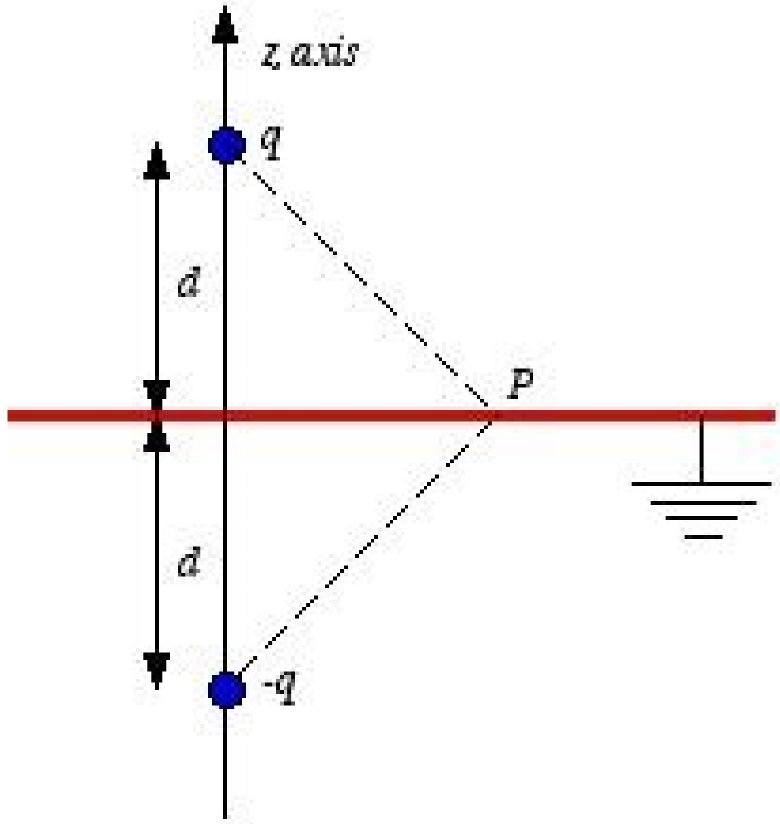
q at $\mathbf{r}_1 = (0, 0, d)$

$-q$ at $\mathbf{r}_2 = (0, 0, -d)$

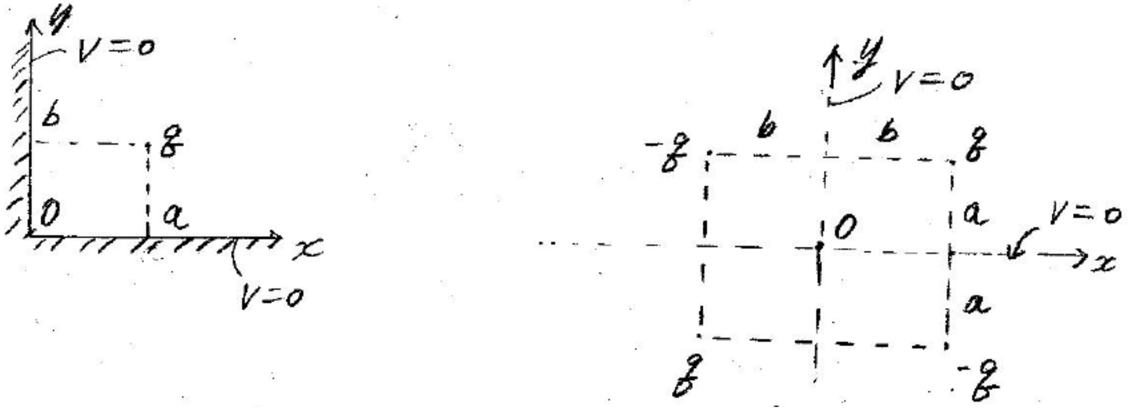
$\mathbf{R}_1 = \mathbf{r} - \mathbf{r}_1,\ \mathbf{R}_2 = \mathbf{r} - \mathbf{r}_2$

$$V(\mathbf{r}) = V_q + V_{-q} = \frac{1}{4\pi\varepsilon}\frac{q}{R_1} - \frac{1}{4\pi\varepsilon}\frac{q}{R_2}$$

$$\mathbf{E} = -\nabla V = \frac{1}{4\pi\varepsilon}\frac{q\hat{\mathbf{R}}_1}{R_1^3} - \frac{1}{4\pi\varepsilon}\frac{q\hat{\mathbf{R}}_2}{R_2^3}$$



- Example: a point source in a right-angle conductor



$\mathbf{r}=(x,y,z)$: field point

q at $\mathbf{r}_1=(a,b,0)$: the original point charge

$-q$ at $\mathbf{r}_2=(-a,b,0)$: an image charge

q at $\mathbf{r}_3=(-a,-b,0)$: an image charge

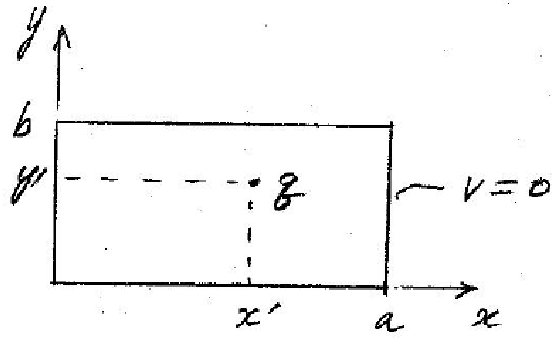
$-q$ at $\mathbf{r}_4=\ (a,-b,0)$: an image charge

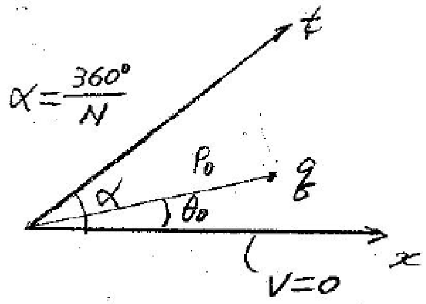
$\mathbf{R}_1=\mathbf{r}-\mathbf{r}_1$, $\mathbf{R}_2=\mathbf{r}-\mathbf{r}_2$, $\mathbf{R}_3=\mathbf{r}-\mathbf{r}_3$, $\mathbf{R}_4=\mathbf{r}-\mathbf{r}_4$

$$V(r)=\frac{q}{4\pi\varepsilon}\left(\frac{1}{R_1}-\frac{1}{R_2}+\frac{1}{R_3}-\frac{1}{R_4}\right)$$

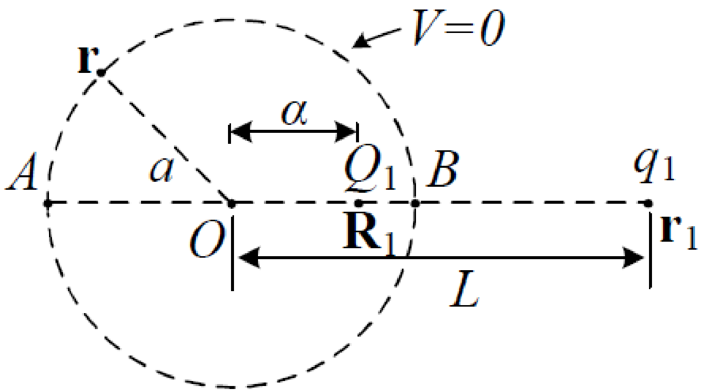
$$\mathbf{E}=-\nabla V=\frac{q}{4\pi\varepsilon}\left(\frac{\hat{\mathbf{R}}_1}{R_1^3}-\frac{\hat{\mathbf{R}}_2}{R_2^3}+\frac{\hat{\mathbf{R}}_3}{R_3^3}-\frac{\hat{\mathbf{R}}_4}{R_4^3}\right)$$

- Example: problems requiring an infinite number of images





- Example: a point charge outside a conducting sphere



$$Q_1 = -\frac{a}{L} q_1$$

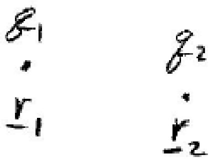
$$\alpha = \frac{a^2}{L}$$

3. Electrostatic Energy

- Energy stored in *N* point charges

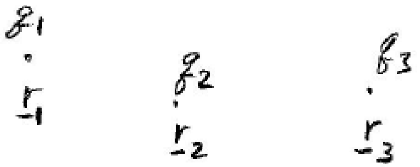
$$q_1 \text{ only}: W = q_1 V_1 = 0 \text{ since no field exists before}$$

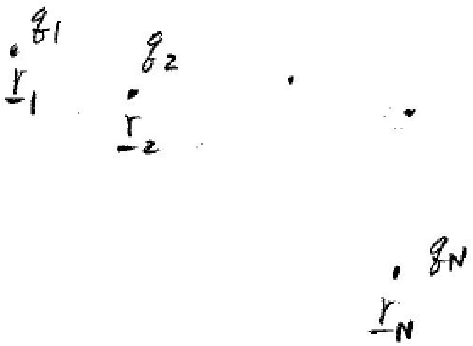
$$q_1, q_2: \; W = W_2 = q_2 V_2 = q_2 \frac{q_1}{4\pi\varepsilon |\mathbf{r}_2 - \mathbf{r}_1|}$$



$$q_1, q_2, q_3: W = W_2 + W_3 = q_2 \frac{q_1}{4\pi\varepsilon |\mathbf{r}_2 - \mathbf{r}_1|} + q_3 \left(\frac{q_1}{4\pi\varepsilon |\mathbf{r}_3 - \mathbf{r}_1|} + \frac{q_2}{4\pi\varepsilon |\mathbf{r}_3 - \mathbf{r}_2|} \right)$$

$$q_1, q_2, q_3, \ldots, q_N: W = W_2 + W_3 + \ldots + W_N = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \left(q_i \sum_{j=1}^{i-1} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \right) = \frac{1}{2} \sum_{i=1}^{N} q_i V_i, \quad V_i = \frac{1}{4\pi\varepsilon} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$





- Energy stored in distributed charges

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i V_i$$

$$q_i = \rho_L dL = \rho_s dS = \rho_v dV$$

$$V_i = \phi(\mathbf{r})$$

$$\sum \rightarrow \int$$

$$W = \frac{1}{2}\int_L \rho_L \phi \, dL = \frac{1}{2}\int_S \rho_s \phi \, dS = \frac{1}{2}\int_V \rho_v \phi \, dV$$

- Electrostatic energy in terms of electric field

$$W = \int_V \frac{1}{2}\rho_v V dV = \int_V \frac{1}{2}(\nabla\cdot\mathbf{D})V dV \quad (\because\ \rho_v = \nabla\cdot\mathbf{D})$$

$$\nabla\cdot(V\,\mathbf{D}) = V(\nabla\cdot\mathbf{D}) + \mathbf{D}\cdot(\nabla V) \rightarrow (\nabla\cdot\mathbf{D})V = \nabla\cdot(V\mathbf{D}) - \mathbf{D}\cdot(\nabla V) = \nabla\cdot(V\mathbf{D}) + \mathbf{D}\cdot\mathbf{E}$$

$$W = \int_V \frac{1}{2}\nabla\cdot(V\mathbf{D})dV + \int_V \frac{1}{2}(\mathbf{D}\cdot\mathbf{E})dV = \oint_S \frac{1}{2}V\,\mathbf{D}\cdot d\mathbf{S} + \int_V \frac{1}{2}(\mathbf{D}\cdot\mathbf{E})dV$$

$$w_s = \frac{1}{2}V\,\mathbf{D}\cdot\mathbf{n} \quad \text{(surface density of electric energy, J/m}^2\text{)}$$

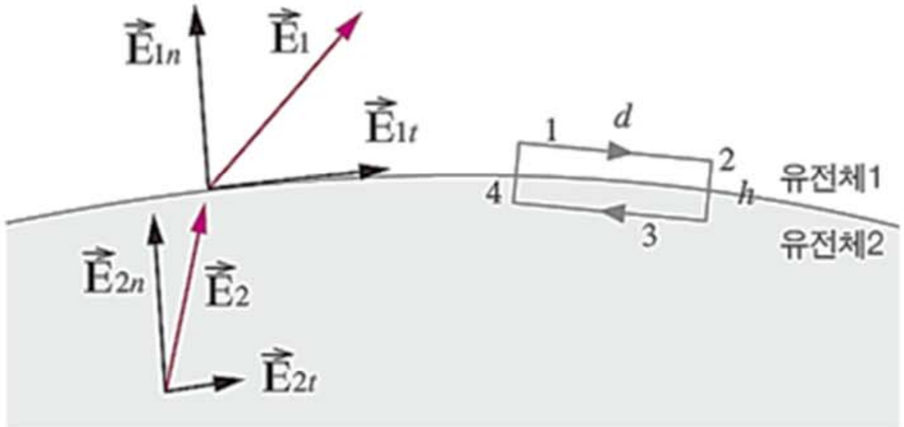
$$w_v = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} \quad \text{(volume density of electric energy, J/m}^3\text{)}$$

$$\oint_S \frac{1}{2}V\,\mathbf{D}\cdot d\mathbf{S} = 0 \text{ as } V \rightarrow \infty$$

$$W = \int_V \frac{1}{2}(\mathbf{D}\cdot\mathbf{E})dV$$

4. Boundary Conditions

- Boundary conditions on a dielectric interface

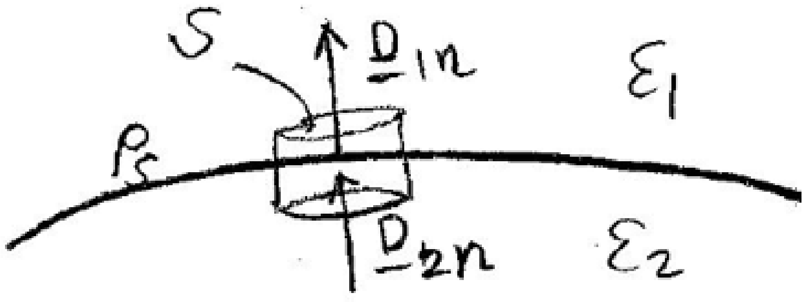


$$\oint \vec{E}\cdot d\vec{l} = \int_1 \vec{E}_{1t}\cdot d\vec{l} + \int_2 \vec{E}_n \cdot d\vec{l} + \int_3 \vec{E}_{2t}\cdot d\vec{l} + \int_4 \vec{E}_n \cdot d\vec{l}$$

$$= E_{1t}d - (E_{1n} + E_{2n})\frac{h}{2} - E_{2t}d + (E_{1n} + E_{2n})\frac{h}{2}$$

$$= (E_{1t} - E_{2t})d = 0$$

$$\longrightarrow \boxed{E_{1t} = E_{2t}}$$



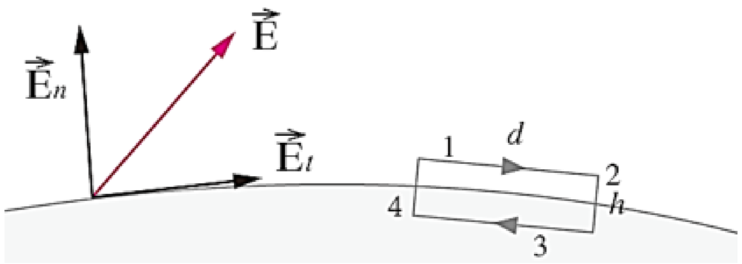
$$\oint D \cdot dS = (D_{1n} - D_{2n})S = Q = \rho_s S$$

$$D_{1n} - D_{2n} = \rho_s$$

$$\frac{E_{1n}}{\varepsilon_1} - \frac{E_{2n}}{\varepsilon_2} = \rho_s$$

$$\rho_s = 0 \rightarrow D_{1n} = D_{2n},\ \frac{E_{1n}}{\varepsilon_1} = \frac{E_{2n}}{\varepsilon_2}$$

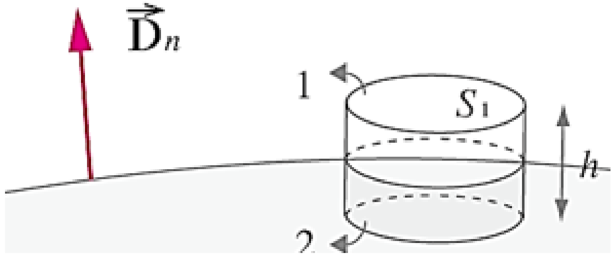
- Boundary conditions on a dielectric-conductor interface



$$\vec{E} = \vec{E}_t + \vec{E}_n$$

$$\oint \vec{E} \cdot d\vec{l} = \int_1 \vec{E}_t \cdot d\vec{l} + \int_2 \vec{E}_n \cdot d\vec{l} + \int_3 \vec{E}_t \cdot d\vec{l} + \int_4 \vec{E}_n \cdot d\vec{l}$$

$$= E_t d - E_n \frac{h}{2} + 0 + E_n \frac{h}{2} = E_t d \longrightarrow \boxed{E_t = 0}$$

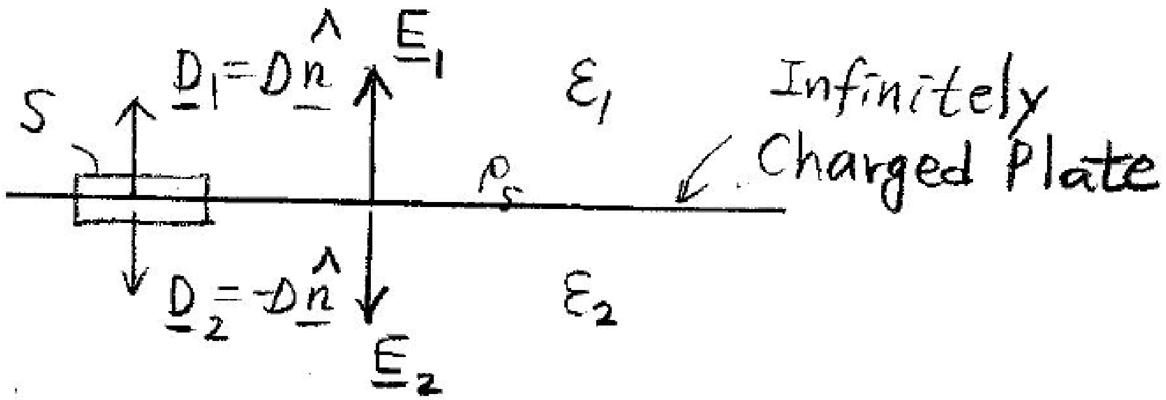


$$\oint \vec{D} \cdot d\vec{S} = \int_{S_1} \vec{D}_n \cdot d\vec{S} + \int_{S_2} \vec{D}_n \cdot d\vec{S} \longrightarrow E_n = \frac{Q}{\varepsilon_0 S} = \frac{\rho_S}{\varepsilon_0} \longrightarrow \boxed{\vec{E} = \frac{\rho_S}{\varepsilon_0}\hat{\mathbf{n}} \ \ (\mathrm{V/m})}$$

$$= \varepsilon_0 E_n S$$

$$= Q$$

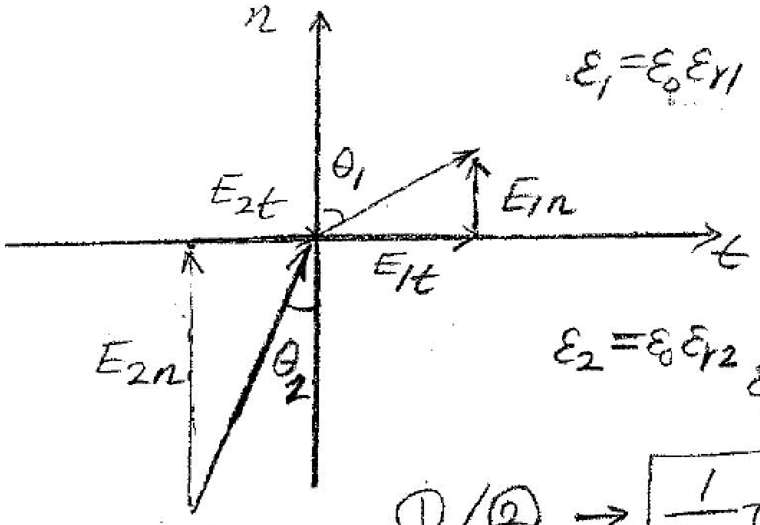
- Example: infinite charged plate



$$\oint \underline{D}\cdot d\underline{S} = Q \rightarrow 2DS = S\rho_s \rightarrow D = \frac{\rho_s}{2}$$

$$\underline{E} = \frac{D}{\varepsilon} \rightarrow \underline{E}_1 = \frac{\rho_s}{2\varepsilon_1}\underline{\hat{n}}, \quad \underline{E}_2 = -\frac{\rho_s}{2\varepsilon_2}\underline{\hat{n}}$$

- Example: angles of electric field lines on a dielectric interface



$\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ $\quad E_{1t} = E_{2t}$

$E_1 \sin\theta_1 = E_2 \sin\theta_2$ ①

$D_{1n} = D_{2n}$

$\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$ $\quad \varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$ ②

$$①/② \rightarrow \boxed{\frac{1}{\varepsilon_1}\tan\theta_1 = \frac{1}{\varepsilon_2}\tan\theta_2}$$

$\varepsilon_1 > \varepsilon_2 \rightarrow \theta_1 > \theta_2$

$$\boxed{E_1 = E_2 \frac{\sin\theta_2}{\sin\theta_1} < E_2}$$

$\varepsilon_{r1} = 80$ (water), $\varepsilon_{r2} = 1$ (air)

$\theta_2 = 15° \longrightarrow \tan\theta_1 = \frac{\varepsilon_1}{\varepsilon_2} = 80 \tan\theta_2$, $\underline{\theta_1 = 87.3°}$

$E_2 = 1\,(V/m) \rightarrow E_1 = E_2 \frac{\sin\theta_2}{\sin\theta_1} = 0.26\,(V/m)$

5. Coding Example

- Example: angles of electric field lines on a dielectric interface

Write down a Python code for the stored energy for *N* point charges.

Input data:

N : number of point charges

qi (C), (xi, yi, zi) m : i-th point charge and position

Output data:

W (J): stored energy

```
# Lecture 7 - Python coding example: Stored energy in n point charges
# Input data:
#  n = number of point charges (maximum 100)
#  qi = charge (C) of the i-th charge
# (xi,yi,zi) = position (m) of the i-th charge
# Output data:
# w = stored energy (J)

from math import *
e0=8.853e-12;pi=3.141593
x=[0]*100;y=[0]*100;z=[0]*100;q=[0]*100
while True:
 n=int(input('Number of point charges='))
 for i in range(n):
  print('For charge no.',i)
  q[i]=float(input('Charge: q(C)='))
  x[i],y[i],z[i]=map(float,input('Position: x,y,z(m)=').split())
 w=0
 for i in range(n):
  sum=0
  for j in range(n):
   if(j == i):
     continue
   else:
     sum=sum+q[j]/sqrt((x[i]-x[j])**2+(y[i]-y[j])**2+(z[i]-z[j])**2)
  w=w+q[i]*sum
 w=w/(4*pi*e0)/2
 print('Stored: W(J)=',w)
```

```
'''
Number of point charges=
3
For charge no. 0
Charge: q(C)=
1e-10
Position: x,y,z(m)=
1 1 3
For charge no. 1
Charge: q(C)=
2e-4
Position: x,y,z(m)=
3 -1 1
For charge no. 2
Charge: q(C)=
4e-10
Position: x,y,z(m)=
4 1 -4
Stored: W(J)= 0.00018318585243
Number of point charges=
'''
```

Fin

(End)