Lecture 3 Terminated Transmission Line

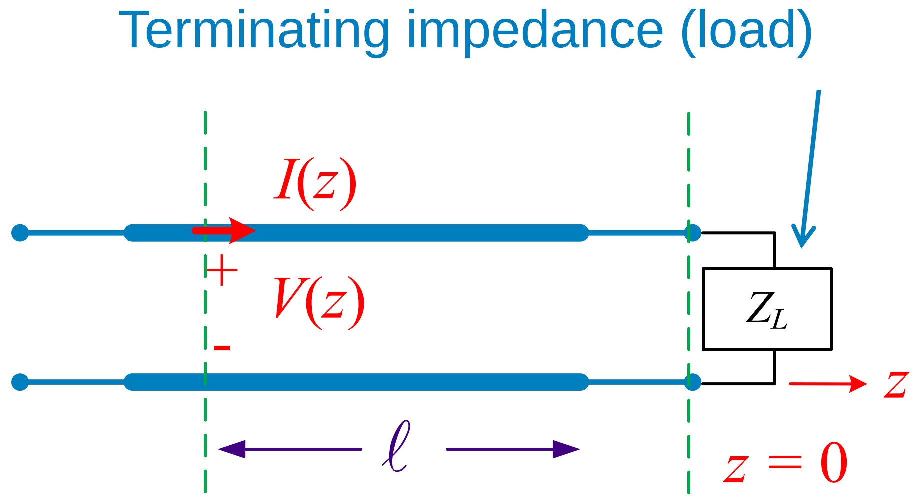
1. Voltage, Current and Input Impedance of A Terminated Line
2. Input Reflection Coefficient and Input Impedance
3. Matched Load, Short-circuit Load and Open-circuit Load
4. Quarter-wave Transformer
5. Voltage Standing Wave Ratio
6. Transmission Line with A Source and A Load
7. Reflected Power and Delivered Power
8. Code Example

1. Voltage, Current and Input Impedance of A Terminated Line

$$\boxed{V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}}$$

Ampl. of voltage wave propagating in positive z direction at $z = 0$.

Ampl. of voltage wave propagating in negative z direction at $z = 0$.

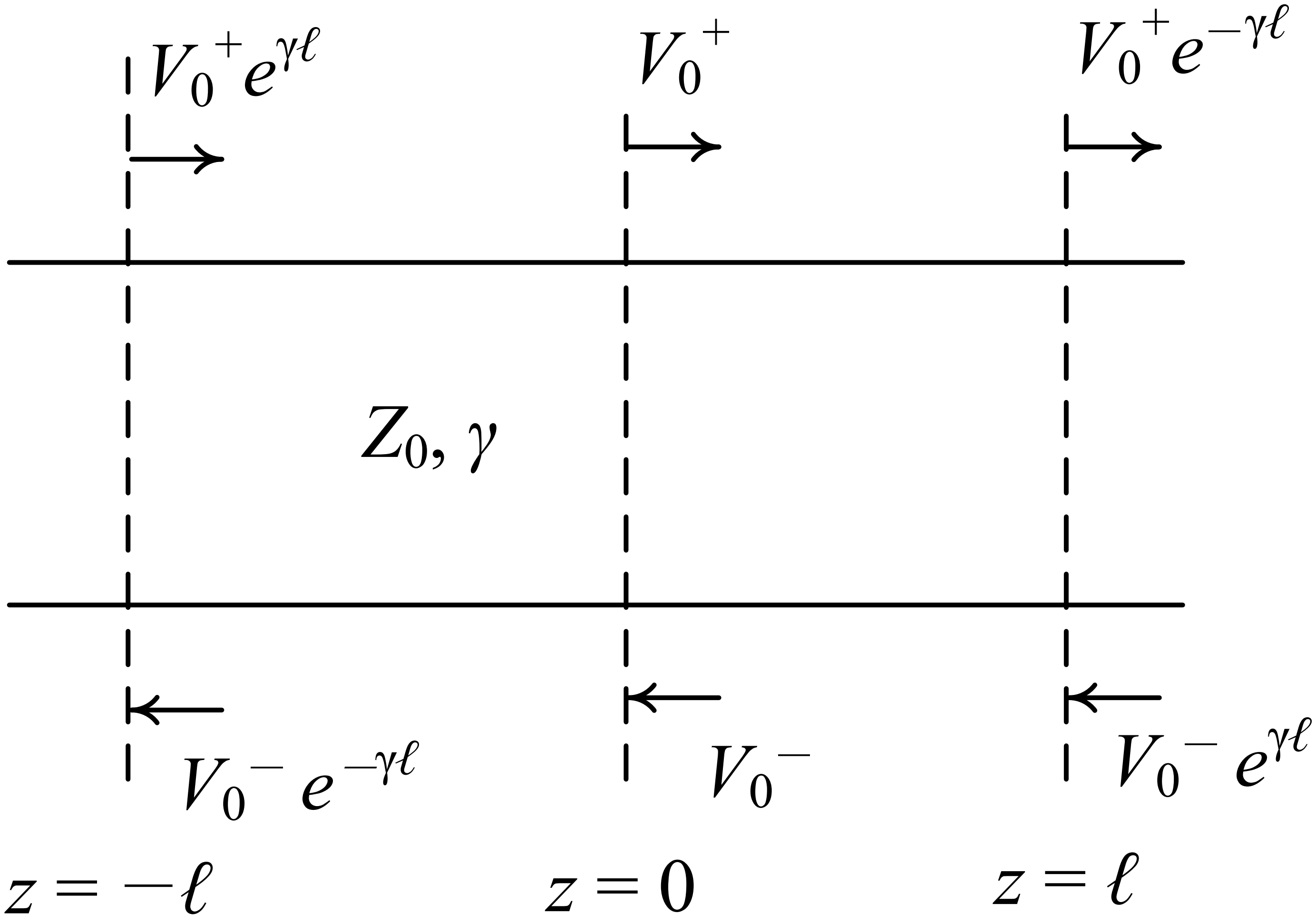


$$V(z) = V^+(z) + V^-(z)$$

$$V^+(z) = V_0^+ e^{-\gamma z}$$

$$V^-(z) = V_0^- e^{\gamma z}$$

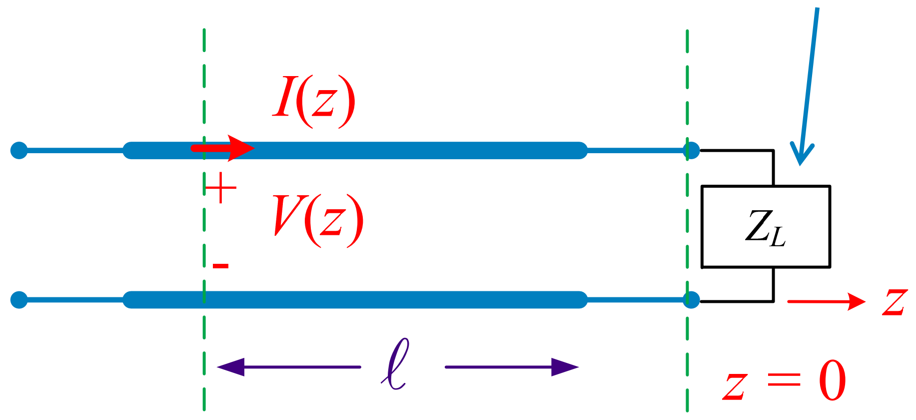
Understanding the wave propagation



Voltage and current:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

Terminating impedance (load)



$\ell \equiv$ distance away from load

What is $V(-\ell)$?

$$\boxed{V(-\ell) = V_0^+ e^{\gamma \ell} + V_0^- e^{-\gamma \ell}}$$

propagating forwards

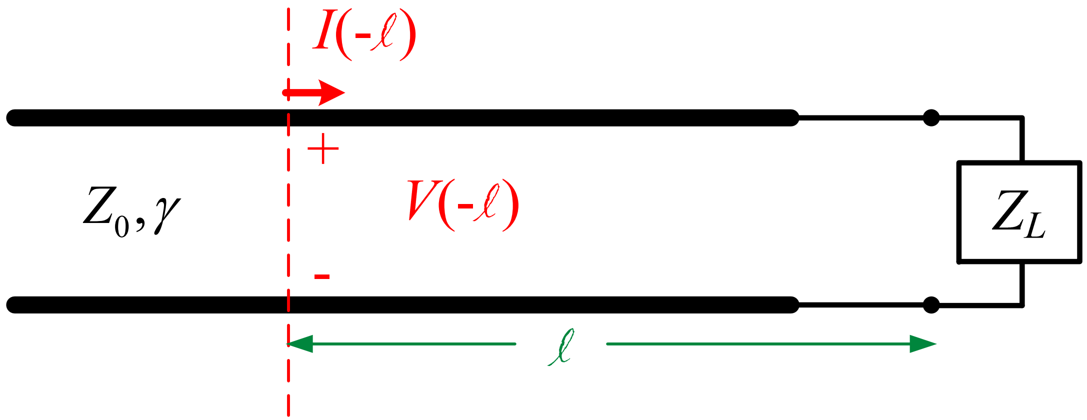
propagating backwards

The current at $z = -\ell$ is then

$$\boxed{I(-\ell) = \frac{V_0^+}{Z_0} e^{\gamma \ell} - \frac{V_0^-}{Z_0} e^{-\gamma \ell}}$$

from $$\frac{dV(z)}{dz} = -(R + j\omega L) I(z),$$

Load reflection coefficient:



Total volt. at distance ℓ from the load

$$V(-\ell) = V_0^+ e^{\gamma\ell} + V_0^- e^{-\gamma\ell} = V_0^+ e^{\gamma\ell}\left(1 + \frac{V_0^-}{V_0^+} e^{-2\gamma\ell}\right)$$

Ampl. of volt. wave prop. towards load, at the load position ($z = 0$).

Ampl. of volt. wave prop. away from load, at the load position ($z = 0$).

$\Gamma_L \equiv$ Load reflection coefficient

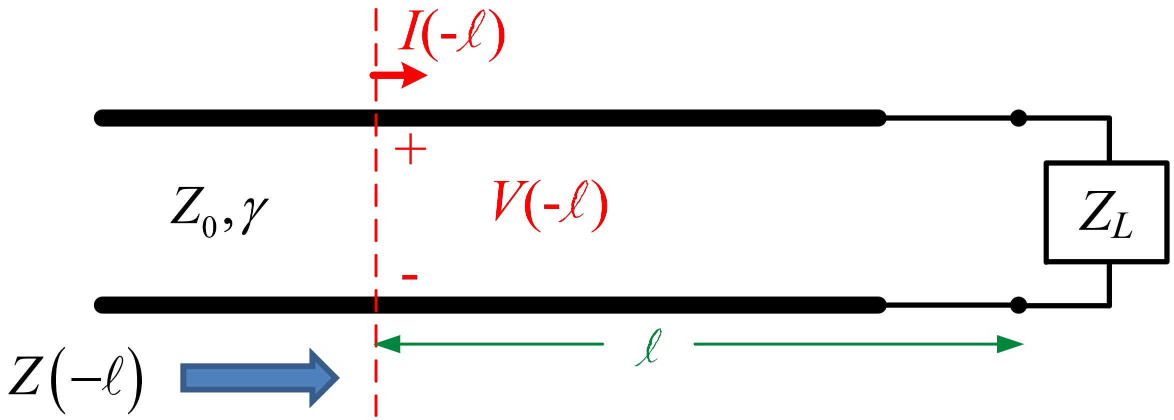
$\Gamma_\ell \equiv$ Reflection coefficient at $z = -\ell$

$$= V_0^+ e^{\gamma\ell}\left(1 + \Gamma_L e^{-2\gamma\ell}\right)$$

Similarly,

$$I(-\ell) = \frac{V_0^+}{Z_0} e^{\gamma\ell}\left(1 - \Gamma_L e^{-2\gamma\ell}\right)$$

Input impedance:



$$V(-\ell) = V_0^+ e^{\gamma\ell}\left(1 + \Gamma_L e^{-2\gamma\ell}\right)$$

$$I(-\ell) = \frac{V_0^+}{Z_0} e^{\gamma\ell}\left(1 - \Gamma_L e^{-2\gamma\ell}\right)$$

$$\boxed{Z(-\ell) = \frac{V(-\ell)}{I(-\ell)} = Z_0\left(\frac{1 + \Gamma_L e^{-2\gamma\ell}}{1 - \Gamma_L e^{-2\gamma\ell}}\right)}$$

Input impedance seen "looking" towards load at $z = -\ell$.

At the load ($\ell = 0$):

$$Z(0) = Z_0\left(\frac{1+\Gamma_L}{1-\Gamma_L}\right) \equiv Z_L$$

$$\Rightarrow \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Reflection Coefficient

$$\Gamma = \Gamma_r + j\Gamma_i$$

$0 \le |\Gamma| \le 1$ if $\mathrm{Re}(Z_L) \ge 0$ (passive load)

$|\Gamma| = 0$ if $Z_L = Z_0$: no reflection

$|\Gamma| = 1$ if $Z_L = 0$ (short), ∞ (open), jX (reactive) : total reflection

$$|\Gamma|\,(\mathrm{dB}) = -20\log_{10}|\Gamma|$$

$$\mathrm{RL}\ (\text{return loss}) = 20\log_{10}|\Gamma|$$

At the load ($\ell = 0$):

$$Z(0) = Z_0\left(\frac{1+\Gamma_L}{1-\Gamma_L}\right) \equiv Z_L \qquad \Rightarrow \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Recall $\quad Z(-\ell) = Z_0\left(\dfrac{1+\Gamma_L e^{-2\gamma\ell}}{1-\Gamma_L e^{-2\gamma\ell}}\right)$

Thus,

$$Z(-\ell) = Z_0\left(\frac{1+\left(\dfrac{Z_L - Z_0}{Z_L + Z_0}\right)e^{-2\gamma\ell}}{1-\left(\dfrac{Z_L - Z_0}{Z_L + Z_0}\right)e^{-2\gamma\ell}}\right)$$

Simplifying, we have

$$Z(-\ell) = Z_0 \left(\frac{1 + \left(\frac{Z_L - Z_0}{Z_L + Z_0} \right) e^{-2\gamma\ell}}{1 - \left(\frac{Z_L - Z_0}{Z_L + Z_0} \right) e^{-2\gamma\ell}} \right) = Z_0 \left(\frac{(Z_L + Z_0) + (Z_L - Z_0) e^{-2\gamma\ell}}{(Z_L + Z_0) - (Z_L - Z_0) e^{-2\gamma\ell}} \right)$$

$$= Z_0 \left(\frac{(Z_L + Z_0) e^{+\gamma\ell} + (Z_L - Z_0) e^{-\gamma\ell}}{(Z_L + Z_0) e^{+\gamma\ell} - (Z_L - Z_0) e^{-\gamma\ell}} \right)$$

$$= Z_0 \left(\frac{Z_L \cosh(\gamma\ell) + Z_0 \sinh(\gamma\ell)}{Z_0 \cosh(\gamma\ell) + Z_L \sinh(\gamma\ell)} \right)$$

sinh: 싸인에이취

Hence, we have

$$\boxed{Z(-\ell) = Z_0 \left(\frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)} \right)}$$

$$\boxed{\gamma = \cancel{\alpha} + j\beta = j\beta}$$

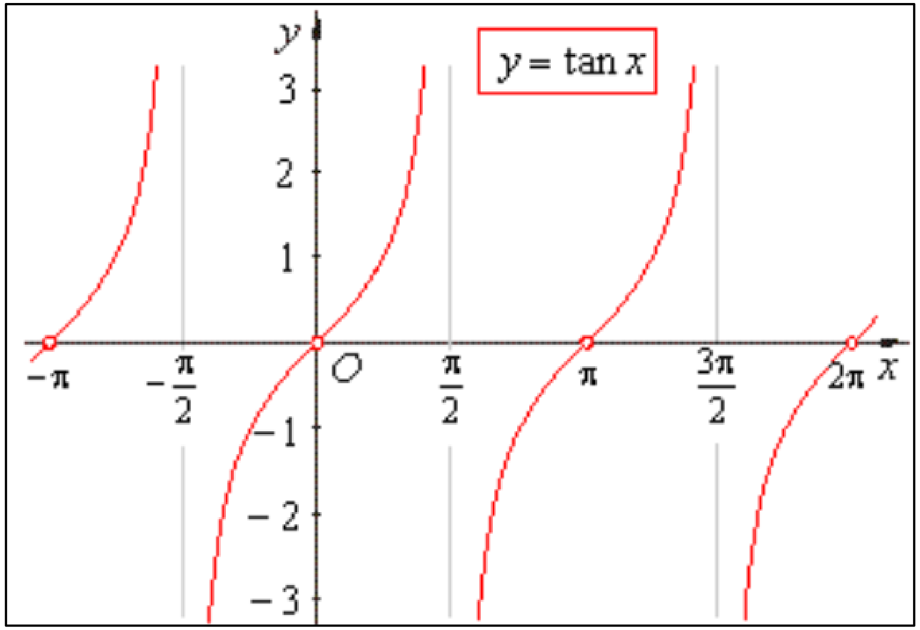
$$V(-\ell) = V_0^+ e^{j\beta\ell}\left(1 + \Gamma_L e^{-2j\beta\ell}\right)$$

$$I(-\ell) = \frac{V_0^+}{Z_0} e^{j\beta\ell}\left(1 - \Gamma_L e^{-2j\beta\ell}\right)$$

$$Z(-\ell) = Z_0\left(\frac{1 + \Gamma_L e^{-2j\beta\ell}}{1 - \Gamma_L e^{-2j\beta\ell}}\right)$$

$$\boxed{Z(-\ell) = Z_0\left(\frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}\right)}$$

Note: $\boxed{\tanh(\gamma\ell) = \tanh(j\beta\ell) = j\tan(\beta\ell)}$



Impedance is periodic with period $\lambda_g/2$

tan repeats when

$$\beta\ell = \pi$$

$$\frac{2\pi}{\lambda_g}\ell = \pi$$

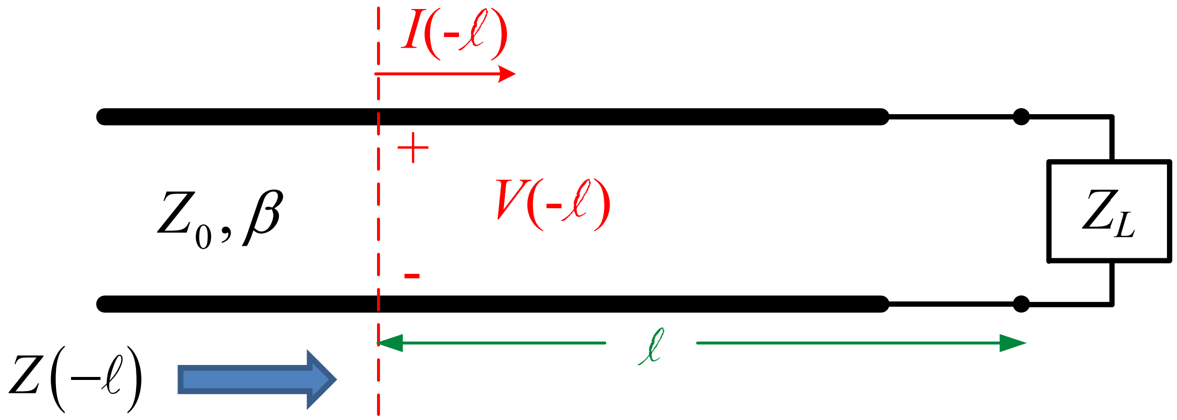
$$\Rightarrow \ell = \lambda_g / 2$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}\ ;\ \cosh x = \frac{e^{x} + e^{-x}}{2}\ ;\ \cosh(jx) = \cos x$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}\ ;\ \sinh x = \frac{e^{x} - e^{-x}}{2}\ ;\ \sinh(jx) = j\sin x$$

$$\tanh x = \frac{\sinh x}{\cosh x}\ ;\ \tanh(jx) = \frac{\sinh(jx)}{\cosh(jx)} = \frac{j\sin x}{\cos x} = j\tan x$$

For the remainder of our transmission line discussion we will assume that the transmission line is lossless.



$$V(-\ell) = V_0^+ e^{j\beta\ell}\left(1+\Gamma_L e^{-2j\beta\ell}\right)$$

$$I(-\ell) = \frac{V_0^+}{Z_0} e^{j\beta\ell}\left(1-\Gamma_L e^{-2j\beta\ell}\right)$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

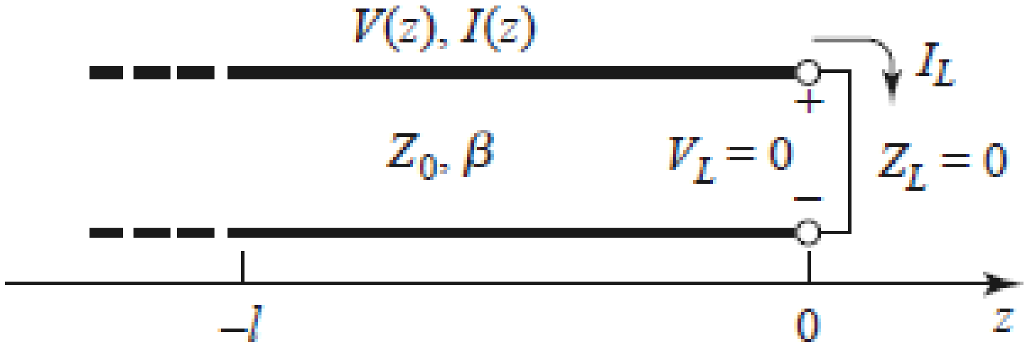
$$\boxed{\lambda_g = \frac{2\pi}{\beta}}$$

$$Z(-\ell) = \frac{V(-\ell)}{I(-\ell)} = Z_0\left(\frac{1+\Gamma_L e^{-2j\beta\ell}}{1-\Gamma_L e^{-2j\beta\ell}}\right)$$

$$= Z_0\left(\frac{Z_L + jZ_0\tan(\beta\ell)}{Z_0 + jZ_L\tan(\beta\ell)}\right)$$

$$v_p = \frac{\omega}{\beta}$$

2. Input Impedance and Input Reflection Coefficient



$$\Gamma_{\text{in}} = \Gamma(-\ell) = \frac{V_0^- e^{-\gamma\ell}}{V_0^+ e^{\gamma\ell}} = \Gamma(0)e^{-2\gamma\ell}$$

$$\Gamma_{\text{in}} = \Gamma(-\ell) = \frac{V_0^- e^{-j\beta\ell}}{V_0^+ e^{j\beta\ell}} = \Gamma(0)e^{-j2\beta\ell}$$

$$V(-\ell) = V_0^+(1+\Gamma_{\text{in}}),\ I(-\ell) = V_0^+(1-\Gamma_{\text{in}})/Z_0$$

$$\boxed{Z_{\text{in}} = \frac{V(-\ell)}{I(-\ell)} = Z_0 \frac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}}}$$

$$\boxed{\Gamma_{\text{in}} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0}}$$

Execise:

The reflection coefficient at the load is given by $\Gamma_L = 0.8\, e^{j45°}$. The line is lossless and has characteristic impedance of 50 Ω.

1) Find the reflection coefficient at $0.2\lambda_g$ away from the load.

2) Find the input impedance at $0.2\lambda_g$ away from the load.

(Solution)

$$1)\ \Gamma_{in} = \Gamma(0)e^{-j2\beta\ell} = 0.8e^{j45°}e^{-j2(2\pi/\lambda_g)(0.2\lambda_g)} = 0.8e^{j45°}e^{-j0.8\pi} = 0.8e^{-j99°}$$

$$2)\ Z_{in} = Z_0\frac{1+\Gamma_{in}}{1-\Gamma_{in}} = 50\frac{1+0.8e^{-j90°}}{1-0.8e^{-j90°}} = 11.0 - j48.8\ \Omega$$

Example: Reflectioin coefficient and input impedance of a terminated transmission line

Transmission line length $L = 0.1\ \lambda_g$; Load impedance $Z_L = 40 + j\ 70\ \Omega$

Characteristic impedance $Z_0 = 50\ \Omega$

1) Find the input reflection coefficient.

2) Fint the input impedance from the reflection coefficient.

3) Find the input impedance using Z_L and L.

(Solution)

$$1)\ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{40 + j70 - 50}{40 + j70 + 50} = 0.620 e^{j60.3^\circ}$$

$$\Gamma_{\text{in}} = \Gamma_L e^{-j2\beta L} = 0.620 e^{j60.3^\circ} e^{-j2(2\pi/\lambda_g)0.1\lambda_g} = 0.620 e^{j60.3^\circ} e^{-j0.4\pi}$$

$$= 0.620 e^{j60.3^\circ} e^{-j72^\circ} = 0.620 e^{-j11.7^\circ}$$

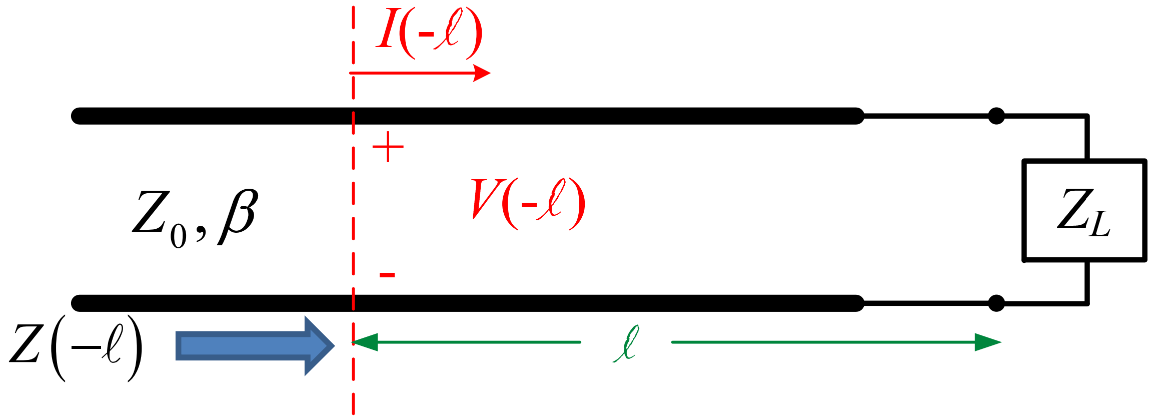
$$2)\ Z_{\text{in}} = Z_0 \frac{1 + \Gamma_{\text{in}}}{1 - \Gamma_{\text{in}}} = 50 \frac{1 + 0.620 e^{-j11.7^\circ}}{1 - 0.620 e^{-j11.7^\circ}} = 180.9 - j73.9\ \Omega$$

$$3)\ Z_{\text{in}} = Z_0 \frac{Z_L + jZ_0 \tan \beta\ell}{Z_0 + jZ_L \tan \beta\ell}$$

$$\beta\ell = (2\pi / \lambda_g)0.1\lambda_g = 0.2\pi\ ; \quad \tan(0.2\pi) = 0.723$$

$$Z_{\text{in}} = 50 \frac{40 + j70 + j50 \times 0.723}{50 + (40 + j70) \times j0.723} = 50 \frac{40 + j106.2}{-0.61 + j28.9} = 182.2 - j73.0\ \Omega$$

3. Matched Load, Short-circuit Load, and Open-circuit Load



Ⓐ Matched load: $(Z_L = Z_0)$

$$\boxed{\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0}$$

No reflection from the load

$$\Rightarrow V(-\ell) = V_0^+ e^{+j\beta\ell}$$

$$I(-\ell) = \frac{V_0^+}{Z_0} e^{+j\beta\ell}$$

$$\boxed{\Rightarrow Z(-\ell) = Z_0}$$

For any ℓ

Short-circuit Load → Inductor

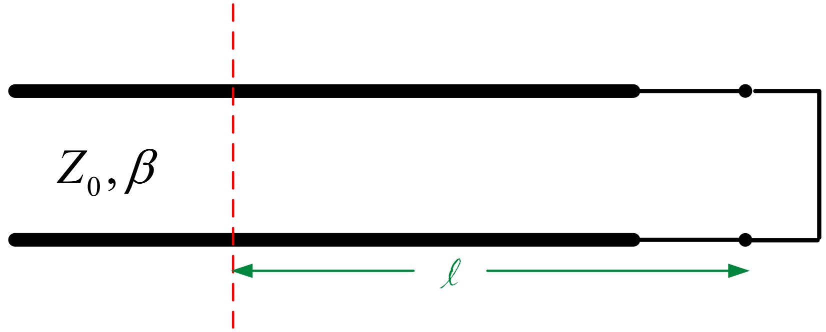
B Short circuit load: ($Z_L = 0$)

$$\Gamma_L = \frac{0 - Z_0}{0 + Z_0} = -1$$

$$\Rightarrow \boxed{Z(-\ell) = jZ_0 \tan(\beta\ell) = j\omega L}$$

Always imaginary!

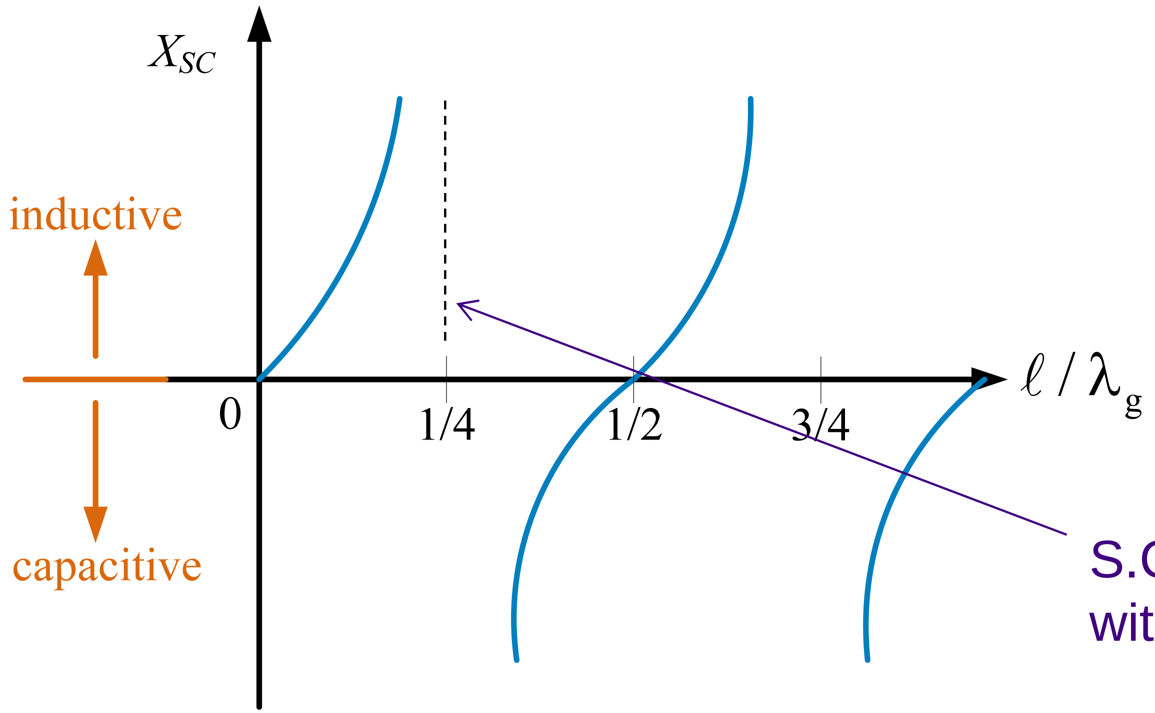
Note: $\beta\ell = 2\pi \dfrac{\ell}{\lambda_g}$



$$\Rightarrow Z(-\ell) = jX_{sc}$$

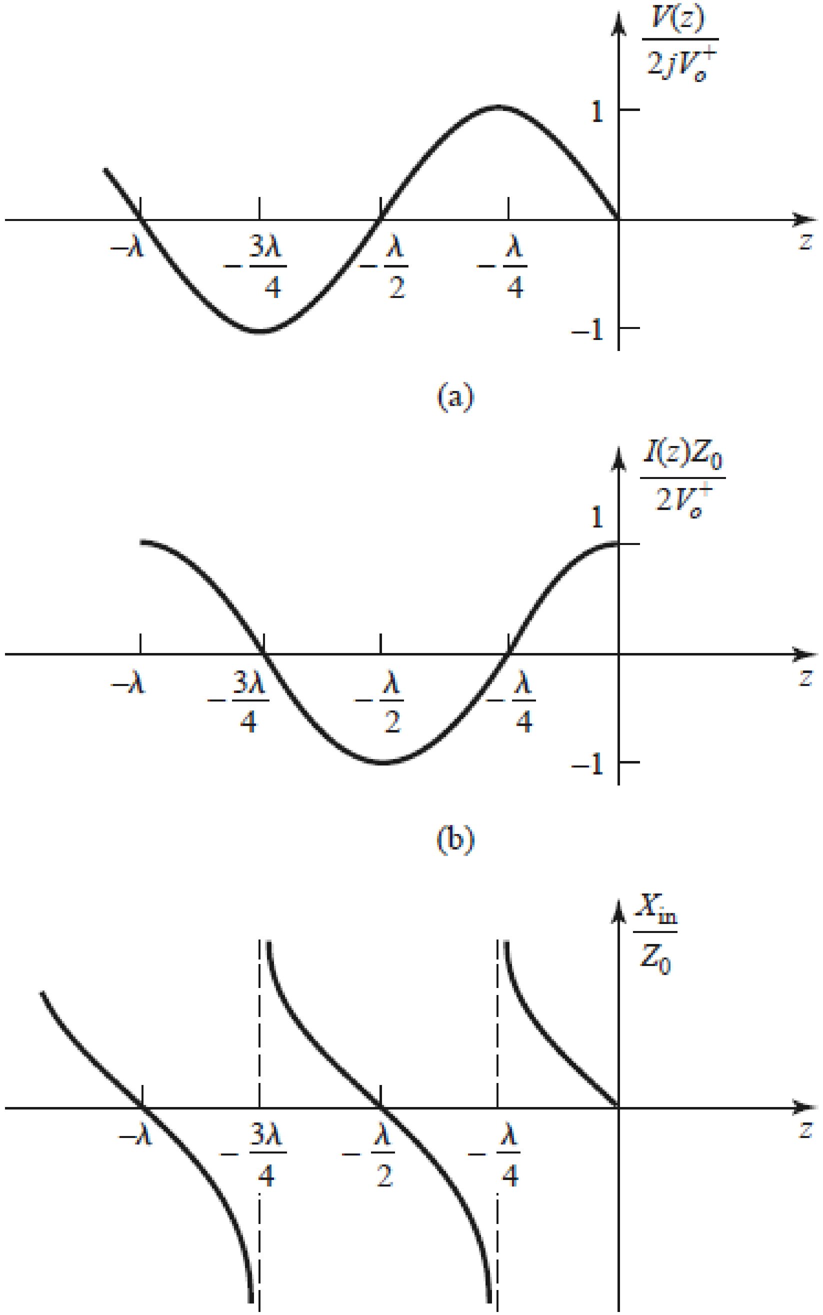
$$X_{sc} = Z_0 \tan(\beta\ell)$$

$$\beta\ell < \pi/2 \rightarrow \boxed{\ell < \lambda_g/4}$$

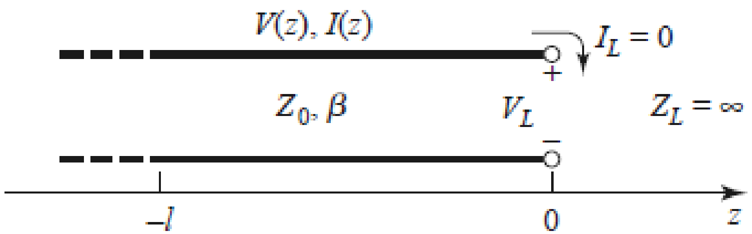


S.C. can become an O.C. with a $\lambda_g/4$ trans. line

Short-circuit Load



Open-circuit Load → Capacitor



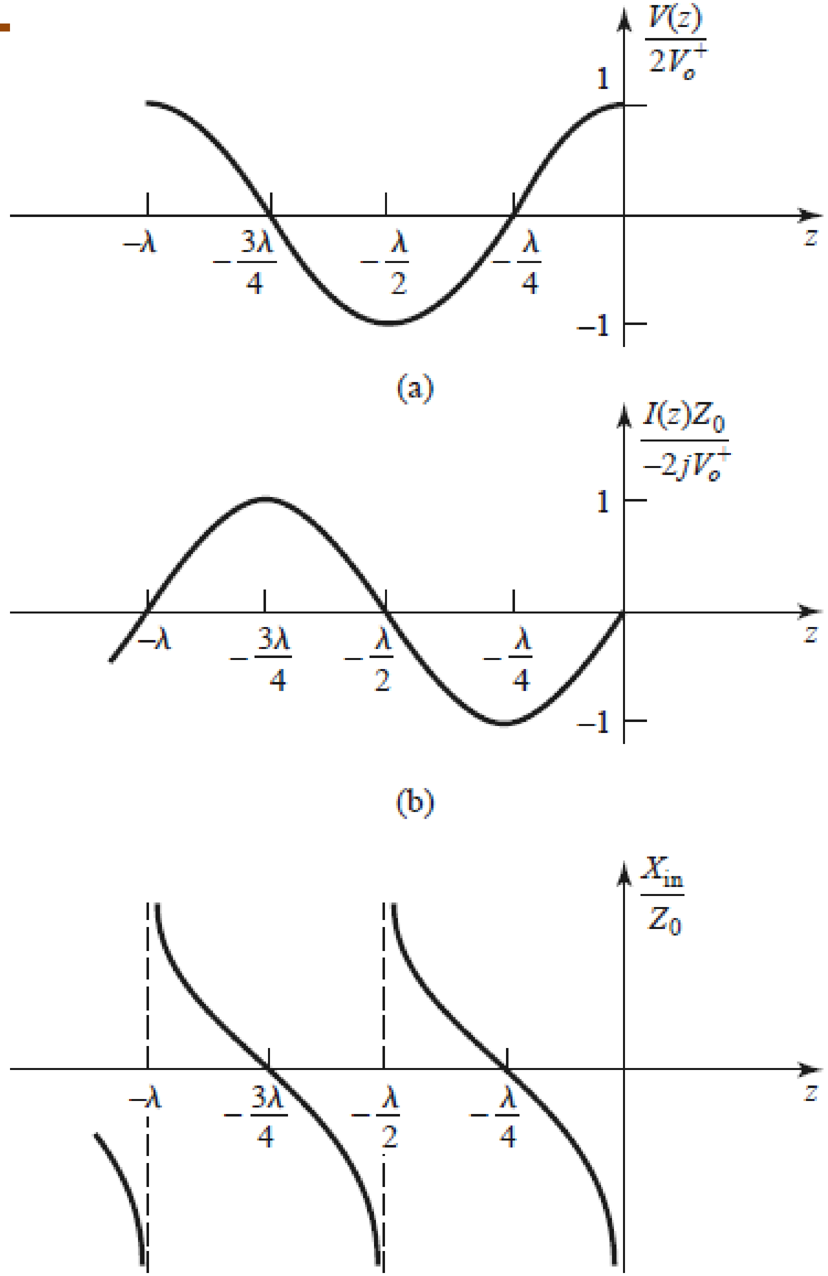
$$Z_L = \infty$$

$$\boxed{\Gamma_L = 1}$$

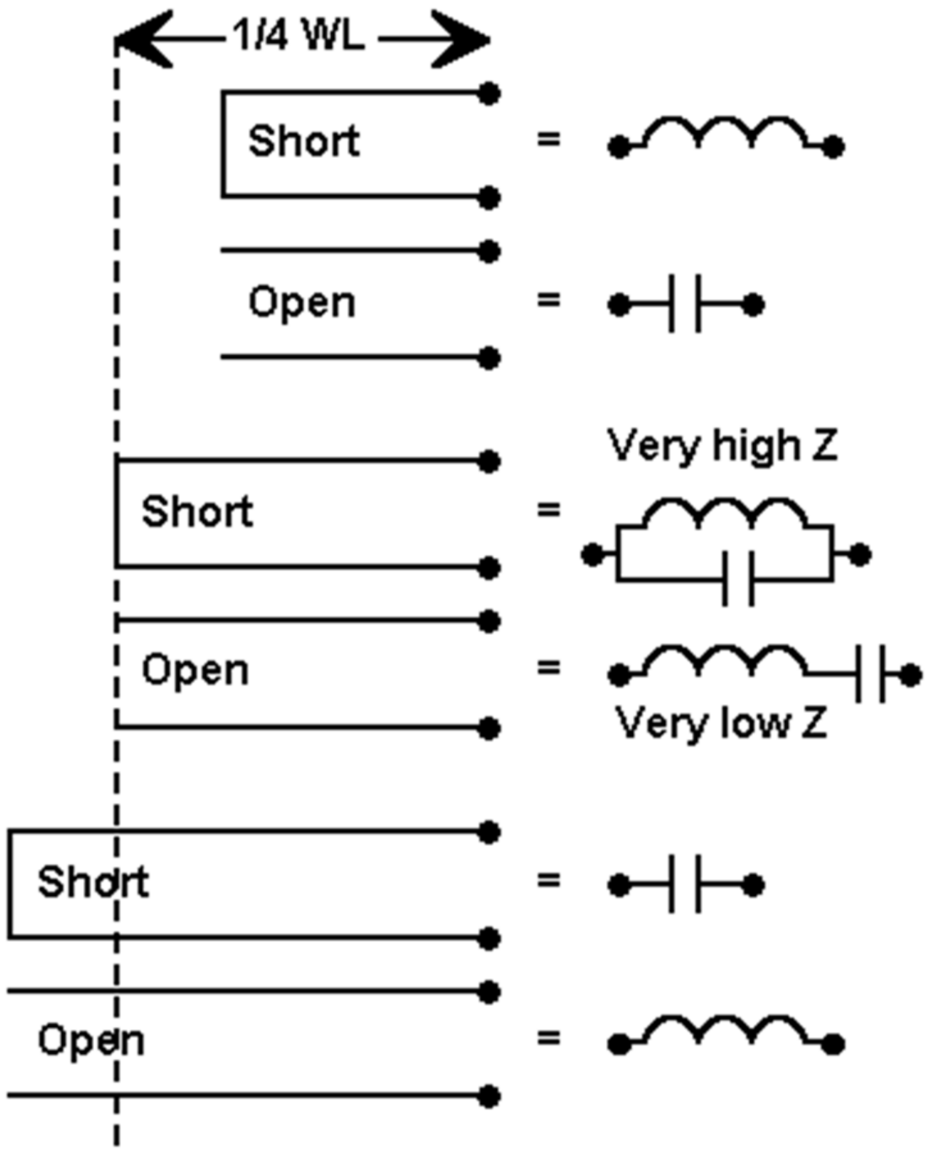
$$Z_{\text{in,short}} = Z_0 \frac{Z_L + jZ_0 \tan \beta\ell}{Z_0 + jZ_L \tan \beta\ell}$$

$$\boxed{= -jZ_0 \cot \beta\ell \equiv -j\frac{1}{\omega C}}$$

$$\beta\ell < \pi/2 \rightarrow \boxed{\ell < \lambda_g/4}$$



Transmission Line Stub Equivalence



Transmission line stub equivalences

Z_0 from Short and Open Circuit Tests

$$Z_{\text{in,short}} = jZ_0 \tan \beta\ell$$

$$Z_{\text{in,open}} = -jZ_0 \cot \beta\ell$$

$$Z_{\text{in,open}} Z_{\text{in,short}} = Z_0^2 \rightarrow \boxed{Z_0 = \sqrt{Z_{\text{in,open}} Z_{\text{in,short}}}}$$

$$-\frac{Z_{\text{in,short}}}{Z_{\text{in,open}}} = \tan^2 \beta\ell \rightarrow \boxed{\beta = \frac{1}{\ell} \tan^{-1} \sqrt{-\frac{Z_{\text{in,short}}}{Z_{\text{in,open}}}}}$$

Example: Short-open test of a transmission line

f = 10 MHz, transmission line length L = 2 m

Open-circuit input impedance $Z_S = j\,36$; Short-circuit input impedance $Z_O = -j\,69$

Find Z_0 and β, λ_g, ε_r the line.

(Answer)

$$Z_0 = \sqrt{Z_S Z_O} = \sqrt{36 \times 69} = 49.8\ \Omega$$

$$\tan \beta L = \sqrt{-\frac{Z_S}{Z_O}} = \sqrt{\frac{36}{69}} = 0.722,\ \beta L = \tan^{-1} 0.722 = 0.625$$

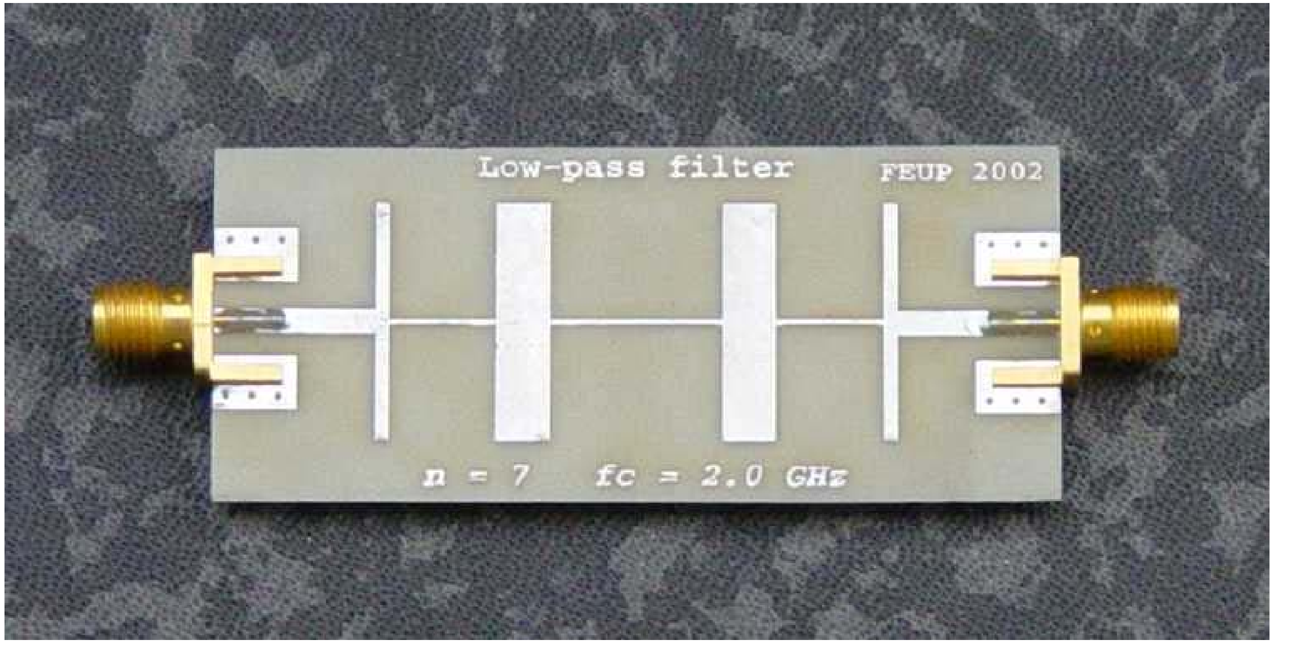
$$\beta = 0.625 / 2 = 0.313 \text{ rad/m}$$

$$\lambda_g = 2\pi / \beta = 20.1 \text{ m}$$

$$\boxed{\lambda_g = \frac{\lambda_0}{\sqrt{\varepsilon_r}}},\ \varepsilon_r = \left(\frac{\lambda_0}{\lambda_g}\right)^2 = (30/20.1)^2 = 2.23$$

Using Transmission Lines to Synthesize Loads

This is very useful is microwave engineering.



A microwave filter constructed from microstrip.

Exercise:

$f = 300$ MHz

Transmission line: length = 0.1 λ_g, Z_0 = 50 Ω

1) Find the equivalent capacitance of an open-circuit transmission line.

2) Find the equivalent inductance of a short-circuit transmission line.

(Solution)

$$\beta L = \frac{2\pi}{\lambda_g} 0.1\lambda_g = 0.1\pi$$

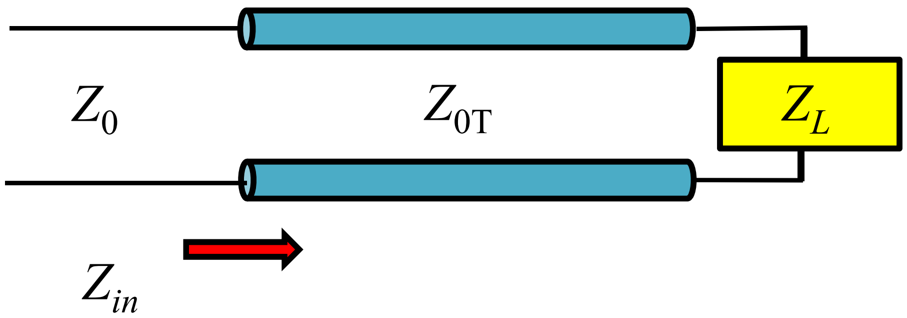
1) $$-jZ_0 \cot \beta L = -j\frac{1}{\omega C}$$

$$C = \frac{1}{(2\pi f)Z_0 \cot \beta L} = \frac{1}{2\pi \times 100 \times 10^6 \times 50 \times \cot(0.1\pi)} = 10.3 \text{ pF}$$

2) $$jZ_0 \tan \beta L = j\omega L$$

$$L = \frac{Z_0 \tan \beta L}{\omega} = \frac{50 \tan(0.1\pi)}{2\pi \times 100 \times 10^6} = 25.9 \text{ nH}$$

4. Quarter-wave Transformer



$$Z_{in} = Z_{0T}\left(\frac{Z_L + jZ_{0T}\tan\beta\ell}{Z_{0T} + jZ_L\tan\beta\ell}\right)$$

$$\boxed{\beta\ell = \beta\frac{\lambda_g}{4} = \frac{2\pi}{\lambda_g}\frac{\lambda_g}{4} = \frac{\pi}{2}}$$

$$\Rightarrow Z_{in} = Z_{0T}\left(\frac{jZ_{0T}}{jZ_L}\right)$$

so

$$\boxed{Z_{in} = \frac{Z_{0T}^2}{Z_L}}$$

$$\Gamma_{in} = 0 \quad \Rightarrow \quad Z_{in} = Z_0$$

$$\Rightarrow \quad Z_0 = \frac{Z_{0T}^2}{Z_L}$$

This requires Z_L to be real.

Hence

$$\boxed{Z_{0T} = \left[Z_0 Z_L\right]^{1/2}}$$

Exercise:

A load with impedance $Z_L = 20\ \Omega$ is connected to a quarter-wave transformer with propagation constant $\beta = 4\pi$ rad/m and characteristic impedance Z_0 for matching with a source impedance of 75 Ω. Find the length and characteristic impedance of the quarter-wave transformer

(Solution)

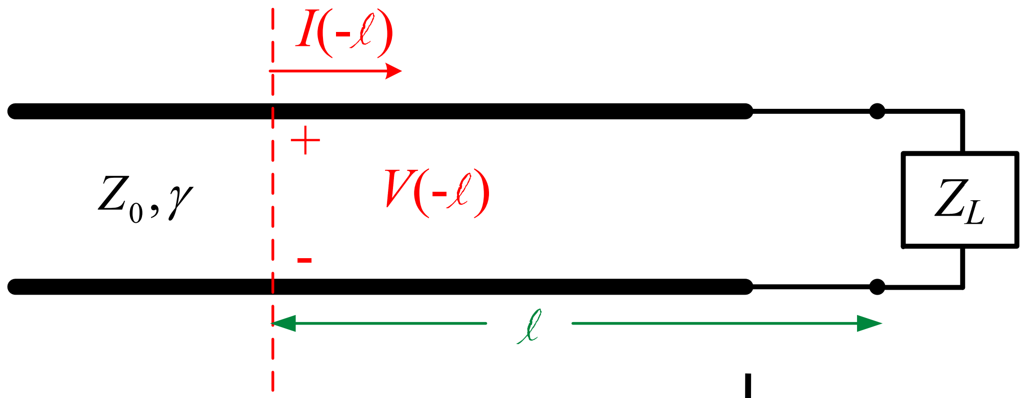
$$\beta = 4\pi = \frac{2\pi}{\lambda_g} \rightarrow \lambda_g = 0.5\ \text{m}$$

$$L = \frac{\lambda_g}{4} = 0.125\ \text{m}$$

$$Z_0 = \sqrt{Z_L Z_S} = \sqrt{20 \times 75} = 38.7\ \Omega$$

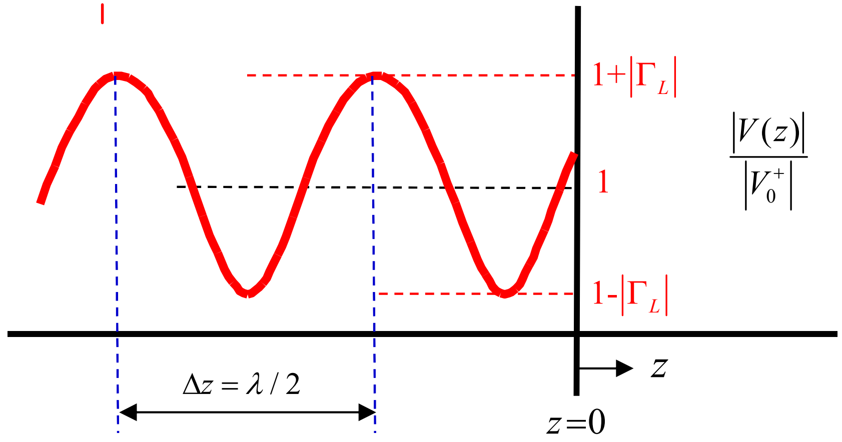
5. Voltage Standing Wave Ratio

$$V(-\ell) = V_0^+ e^{j\beta\ell}\left(1+\Gamma_L e^{-2j\beta\ell}\right)$$
$$= V_0^+ e^{j\beta\ell}\left(1+\left|\Gamma_L\right| e^{j\phi_L} e^{-2j\beta\ell}\right)$$



$$\left|V(-\ell)\right| = \left|V_0^+\right|\left|1+\left|\Gamma_L\right| e^{j\phi_L} e^{-j2\beta\ell}\right|$$

$$V_{\max} = \left|V_0^+\right|\left(1+\left|\Gamma_L\right|\right),\ \phi_L - 2\beta\ell = 2n\pi$$
$$V_{\min} = \left|V_0^+\right|\left(1-\left|\Gamma_L\right|\right),\ \phi_L - 2\beta\ell = 2n\pi + \pi$$



$$\boxed{\text{Voltage Standing Wave Ratio (VSWR)} = \frac{V_{\max}}{V_{\min}}}$$

$$\boxed{\text{VSWR} = \frac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|}}$$

Exercise:

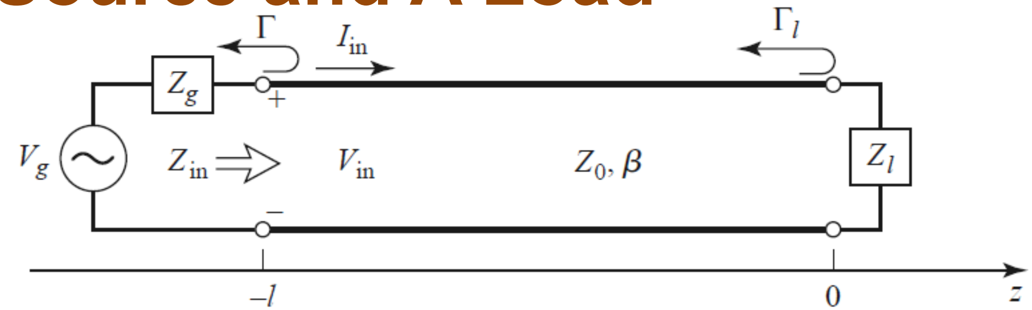
A load with impedance $Z_L = 20 - j\,20\ \Omega$ is connected to a transmission line with characteristic impedance $Z_0 = 50\ \Omega$. Find the VSWR on the transmission line.

(Solution)

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{20 - j20 - 50}{20 - j20 + 50} = \frac{-30 - j20}{70 - j20} = 0.495e^{-j130.4^\circ}$$

$$\text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1 + 0.495}{1 - 0.495} = 2.96$$

6. Transmission Line with A Source and A Load



$$Z_{in} = Z_0 \frac{1 + \Gamma_\ell e^{-2j\beta\ell}}{1 - \Gamma_\ell e^{-2j\beta\ell}} = Z_0 \frac{Z_\ell + jZ_0 \tan\beta\ell}{Z_0 + jZ_\ell \tan\beta\ell}, \tag{2.67}$$

where Γ_ℓ is the reflection coefficient of the load:

$$\Gamma_\ell = \frac{Z_\ell - Z_0}{Z_\ell + Z_0}. \tag{2.68}$$

The voltage on the line can be written as

$$V(z) = V_o^+ \left(e^{-j\beta z} + \Gamma_\ell e^{j\beta z}\right), \tag{2.69}$$

and we can find V_o^+ from the voltage at the generator end of the line, where $z = -\ell$:

$$V(-\ell) = V_g \frac{Z_{in}}{Z_{in} + Z_g} = V_o^+ \left(e^{j\beta\ell} + \Gamma_\ell e^{-j\beta\ell}\right),$$

so that

$$V_o^+ = V_g \frac{Z_{in}}{Z_{in} + Z_g} \frac{1}{\left(e^{j\beta\ell} + \Gamma_\ell e^{-j\beta\ell}\right)}. \tag{2.70}$$

This can be rewritten, using (2.67), as

$$V_o^+ = V_g \frac{Z_0}{Z_0 + Z_g} \frac{e^{-j\beta\ell}}{\left(1 - \Gamma_\ell \Gamma_g e^{-2j\beta\ell}\right)}, \tag{2.71}$$

Exercise:

A load with impedance $Z_L = 20 - j\,20\ \Omega$ is connected to a transmission line whose length is 0.2 wavelength and whose characteristic impedance Z_0 is 50 Ω. The other end of the transmission line is connected to a voltage source $V_s = 10e^{j45°}$ with internal impedance Z_s of 40+j60 Ω.

1) Find the voltage at the input of the transmission line.

2) Find the voltage at the load.

3) Find the power dissipated at the load.

(Solution)

$$1)\ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{20 - j20 - 50}{20 - j20 + 50} = \frac{36.1e^{j213.7°}}{72.8e^{-j15.9°}} = 0.496e^{j229.6°}$$

$$\Gamma_{\text{in}} = \Gamma_L e^{-j2(2\pi/\lambda_g)0.2\lambda_g} = 0.496e^{j229.6°}e^{-j0.8\pi} = 0.496e^{j85.6°} = 0.0381 + j0.495$$

$$Z_{\text{in}} = Z_0\frac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}} = 50\frac{1+0.0381+j0.495}{1-0.0381-j0.495} = 50\frac{1.0381+j0.495}{0.962-j0.495} = 32.2 + j42.3\ \Omega$$

$$V_{\text{in}} = V_s\frac{Z_{\text{in}}}{Z_s + Z_{\text{in}}} = 10e^{j45°}\frac{32.2+j42.3}{40+j60+32.2+j42.3} = 4.25e^{j42.9°}$$

2) $V_{\text{in}} = V_{0,\text{in}}^{+}(1+\Gamma_{\text{in}})$

$$V_{0,\text{in}}^{+} = \frac{V_{\text{in}}}{1+\Gamma_{\text{in}}} = \frac{4.25e^{j42.9^\circ}}{1+0.496e^{j85.6^\circ}} = 3.70e^{j17.4^\circ}$$

$$V_L = V_{0,\text{in}}^{+} e^{-j(2\pi/\lambda_g)0.2\lambda_g}(1+\Gamma_L) = 3.70e^{j17.4^\circ}e^{-j0.4\pi}(1+0.496e^{j229.6^\circ}) = 2.87e^{-j83.7^\circ}$$

3) $P_L = \frac{1}{2}\text{Re}\frac{|V_L|}{Z_L^*} = \frac{1}{2}\text{Re}\frac{2.87^2}{(20-j20)^*} = \frac{1}{2}\times 0.103 = 0.0515\ \text{W}$

7. Reflected Power and Delivered Power

$$V(z) = V_o^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right),$$

$$I(z) = \frac{V_o^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right).$$

$$P_{\text{avg}} = \frac{1}{2}\text{Re}\left\{V(z)I(z)^*\right\} = \frac{1}{2}\frac{|V_o^+|^2}{Z_0}\text{Re}\left\{1 - \Gamma^* e^{-2j\beta z} + \Gamma e^{2j\beta z} - |\Gamma|^2\right\},$$

$$P_{\text{avg}} = \frac{1}{2}\frac{|V_o^+|^2}{Z_0}\left(1 - |\Gamma|^2\right),$$

$$\boxed{\begin{aligned} &P_i = \frac{1}{2}\frac{|V_0^+|^2}{Z_0} \quad \text{: incident power} \\ &P_r = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}|\Gamma|^2 \quad \text{: reflected power} \\ &P_{\text{avg}} = P_i - P_r \quad \text{: power delivered to the load} \end{aligned}}$$

Exercise:

A load with impedance $Z_L = 20 - j\,20\ \Omega$ is connected to a transmission line with characteristic impedance $Z_0 = 50\ \Omega$. A voltage of $20e^{j45^\circ}$ is incident on the load.

1) Find the power P_i incident on the load.

2) Find the power P_r reflected from the load.

3) Find the power P_L delivered to the load.

(Solution)

1) $P_i = \dfrac{1}{2}\dfrac{|V_0^+|^2}{Z_0} = \dfrac{1}{2}\dfrac{20^2}{50} = 4\ \text{W}$

2) $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{20 - j20 - 50}{20 - j20 + 50} = \dfrac{-30 - j20}{70 - j20},\ |\Gamma_L|^2 = \dfrac{30^2 + 20^2}{70^2 + 20^2} = \dfrac{13}{53}$

$$P_r = |\Gamma_L|^2 P_i = \frac{13}{53} \times 4 = \frac{52}{53} = 0.98\ \text{W}$$

3) $P_L = P_i - P_r = 4 - 0.98 = 3.02\ \text{W}$

8. Code Example

Given,

$Z_L = R_L + jX_L$: load impedance (Ω)

Z_0 : characteristic impedance (Ω)

L : line length in wavelength

Find

$\Gamma_{\text{in}} = |\Gamma_{\text{in}}| e^{j\varphi}$: input reflection coefficient in polar form

$|\Gamma_{\text{in}}|$ (dB) : reflection coefficient in dB

$Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}}$: input impedance

(Equations)

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}, \ \Gamma_{\text{in}} = \Gamma_L e^{-j2\beta L} = \Gamma_L e^{-j2(2\pi/\lambda)L} = \Gamma_L e^{-j2\times 2\pi(L/\lambda)}$$

$$|\Gamma_{\text{in}}|\,(\text{dB}) = 20\log_{10}|\Gamma_{\text{in}}|, \ Z_{\text{in}} = Z_0 \frac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}}$$

```
# Microwave Engingineering, Lecture 2 Python Code
# Terminated transmission line: input reflection coefficient and input impedance
# Sample values:
# Input: RL=40, XL=40, L=0.1 wavelenth
# Output:
#
import cmath
import math
pi=3.14159265
while True:
 RL=float(input('RL (ohm)='))
 XL=float(input('XL (ohm)='))
 Z0=float(input('Z0 (ohm)='))
 L=float(input('L (wavelength)='))
 j=complex(0.,1.)
 ZL=complex(RL, XL)
 GL=(ZL-Z0)/(ZL+Z0)
 Gin=GL*cmath.exp(-j*2*2*pi*L)
 Gin_mag, Gin_phase=cmath.polar(Gin)
 Gin_dB=20*math.log(Gin_mag)
 Zin=Z0*(1+Gin)/(1-Gin)
 print('Input reflection coeffcient: mag, phase(deg)=',Gin_mag, Gin_phase)
 print('Input reflection coefficint (dB)=',Gin_dB)
 print('Zin=',Zin)
```

```
'''
RL (ohm)=
20
XL (ohm)=
50
Z0 (ohm)=
50
L (wavelength)=
0.2
Input reflection coeffcient: mag, phase(deg)= 0.677834389404565 -1.022307778917341
Input reflection coefficint (dB)= -7.7770456858800845
Zin= (35.91076216402567-76.8526053239948j)
RL (ohm)=
'''
```

Fin

(End)