Lecture 11 Maxwell's Equations

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1. Maxwell's Equations

- Maxwell's equations (ME)

$$\nabla \cdot \mathbf{D} = \rho \quad \text{(Gauss's law)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law)}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss's law)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{(Ampere's law)}$$

- ME in terms of **E** and **H**

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{(Faraday's law)}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \text{(Gauss's law)}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere's law)}$$

- Materials electrical properties

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \text{(dielectric property)}$$

$$\mathbf{B} = \mu \mathbf{H} \quad \text{(magnetic property)}$$

$$\mathbf{J} = \sigma \mathbf{E} \quad \text{(conductive property)}$$

- Sinusoidal field

$$\boldsymbol{\mathcal{E}} = \mathbf{E}_0 \cos(\omega t + \phi_e)$$

$$\boldsymbol{\mathcal{H}} = \mathbf{H}_0 \cos(\omega t + \phi_h)$$

$$\frac{\partial}{\partial t} \rightarrow j\omega \quad \text{and} \quad \int dt \rightarrow \frac{1}{j\omega}$$

$\mathbf{E} = \mathbf{E}_0 e^{j\phi_e}$: electric field phasor

$\mathbf{H} = \mathbf{H}_0 e^{j\phi_h}$: magnetic field phasor

- ME in phasor form

$$\boxed{\begin{aligned} &\nabla \cdot \mathbf{E} = \rho / \varepsilon \\ &\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \\ &\nabla \cdot \mathbf{H} = 0 \\ &\nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\mathbf{E} \end{aligned}}$$

2. Power and Energy in Electromagnetic Fields

- Power in terms of voltage and current

DC:

$$W = \frac{1}{2}VQ = \frac{1}{2}\frac{Q^2}{C} \quad \text{(dc capacitor stored energy)}$$

$$P = \frac{dW}{dt} = \frac{1}{2}\frac{2Q}{C}\frac{dQ}{dt} = VI \quad \text{(dc capacitor power)}$$

AC:

$$v(t) = V_0 \cos(\omega t + \phi_v), \;\; i(t) = I_0 \cos(\omega t + \phi_i)$$

$$V = V_0 e^{j\phi_v}, \; I = I_0 e^{j\phi_i}$$

$$p(t) = vi = \frac{1}{2}V_0 I_0 \left[\cos(\phi_v - \phi_i) + \cos(2\omega t + \phi_v + \phi_i)\right]$$

$$P_{\text{dc}} = \frac{1}{2}V_0 I_0$$

$$P_{\text{ac}} = \frac{1}{2}V_0 I_0 \cos(2\omega t + \phi_v + \phi_i)$$

$$P_{\text{av}} = \langle p \rangle = \frac{1}{T}\int_0^T p(\tau)d\tau = \frac{1}{2}V_0 I_0 \cos(\phi_v - \phi_i)$$

$$P_{\text{av}} = \frac{1}{2}\text{Re}(VI^*) = \frac{1}{2}\text{Re}(V^* I)$$

$$\boxed{P = \frac{1}{2}VI^* = \frac{1}{2}V^* I \;\; \text{(complex power)}}$$

- Power loss in dielectric materials

$$w=\frac{1}{2}vq=\frac{1}{2}\frac{q^2}{C} \quad \text{(capacitor stored energy)}$$

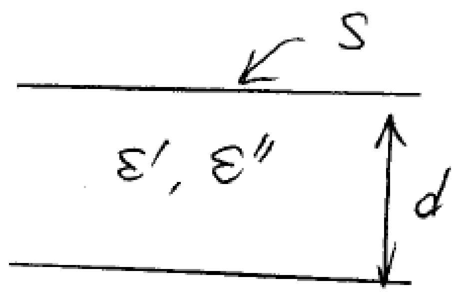
$$p=vi,\ i=C\frac{dv}{dt}\rightarrow P=\frac{1}{2}V^*I=\frac{1}{2}V^*(j\omega CV)=\frac{1}{2}(j\omega)C|V|^2$$

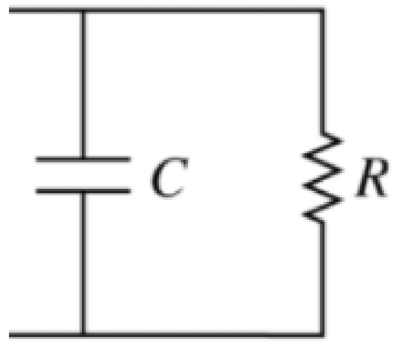
$$C=(\varepsilon'-j\varepsilon'')\frac{S}{d}$$

$$P=\frac{1}{2}(\omega\varepsilon'')\frac{S}{d}|V|^2+j\frac{1}{2}(\omega\varepsilon')\frac{S}{d}|V|^2$$

$$P=\frac{1}{2}\frac{|V|^2}{R}+j\omega\frac{1}{2}C|V|^2=\frac{1}{2}Y|V|^2$$

$$Y=\frac{1}{R}+j\omega C$$





$$\boxed{R=\frac{d}{(\omega\varepsilon'')S}=\frac{d}{\sigma S}} \quad \text{(resistance in parallel with capacitance)}$$

$$\boxed{C=\varepsilon'\frac{S}{d}} \quad \text{(capacitance)}$$

- Complex permittivity and electric loss tangent

$\boxed{\varepsilon = \varepsilon' - j\varepsilon''}$ (complex permittivity)

$\boxed{\tan\delta = \dfrac{\varepsilon''}{\varepsilon'}}$ (loss tangent)

$\varepsilon' = \varepsilon_0 \varepsilon_r$: accounts for dielectric polarization

ε'' : accounts for dielectric loss

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m (permittivity of vacuum)

ε_r : relative permittivity = dielectric constant

ε', ε'': materials with dielectric polarization and dielectric loss

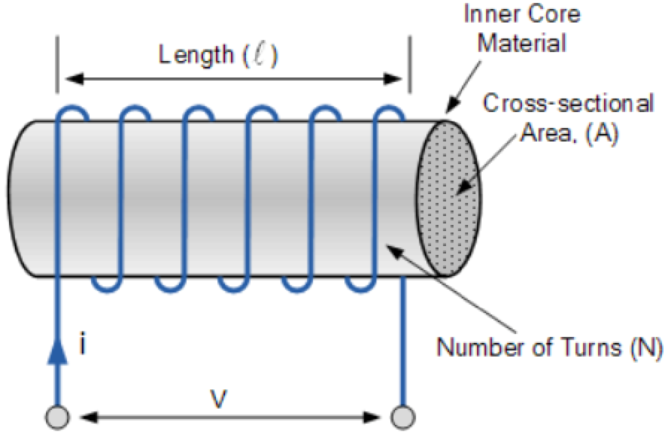
ε', $\boxed{\sigma = \omega\varepsilon''}$: materials with dielectric polarization and conduction loss

- Power loss in magnetic materials

$$w = \frac{1}{2} i\phi = \frac{1}{2} L i^2 \quad \text{(magnetic energy stored in an inductor)}$$

$$p = vi,\ v = L\frac{di}{dt} \rightarrow P = \frac{1}{2} I^* V = \frac{1}{2} I^* (j\omega L I) = \frac{1}{2}(j\omega) L |I|^2$$

$$L = (\mu' - j\mu'')\frac{A}{\ell} \quad \text{(inductance of a torroidal inductor)}$$

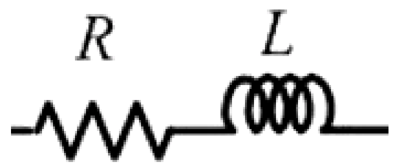


$$P = \frac{1}{2}(\omega\mu'')\frac{A}{\ell}|I|^2 + j\frac{1}{2}(\omega\mu')\frac{A}{\ell}|I|^2$$

$$P = \frac{1}{2} R |I|^2 + j\omega \frac{1}{2} L |I|^2 = \frac{1}{2} Z |V|^2$$

$$Z = R + j\omega L$$

$$\boxed{R = \omega\mu'' \frac{NA}{\ell}} \quad \text{(resistor in series with a lossless inductor)}$$



$$\boxed{L = \mu' \frac{N^2 A}{\ell}} \quad \text{(inductance)}$$

- Complex permeability and magnetic loss tangent

$\boxed{\mu = \mu' - j\mu''}$ (complex permeability)

$\boxed{\tan\delta_m = \dfrac{\mu''}{\mu'}}$ (magnetic loss tangent)

$\boxed{\mu' = \mu_0 \mu_r}$: accounts for magnetization of a material

μ'' : accounts for magnetic loss of a material

$\mu_0 = 4\pi \times 10^{-7}$ H/m (permeability of vacuum)

μ_r : relative permeability

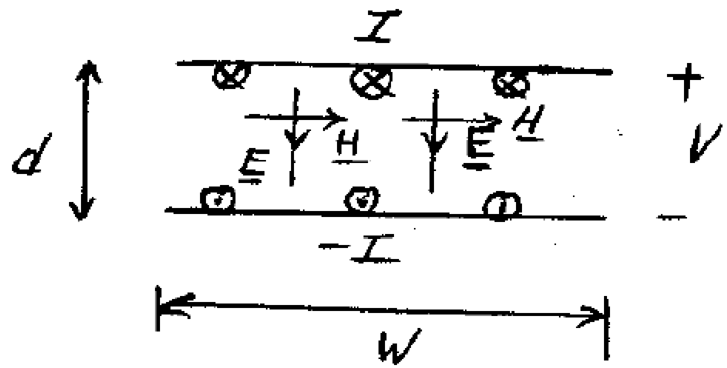
- Power density of electromagnetic fields

- Use an ideal parallel-plate capacitor to understand the concept

$$V = Ed$$

$$I = WH$$

$$P = \frac{1}{2} I^* V = \frac{1}{2} V^* I$$

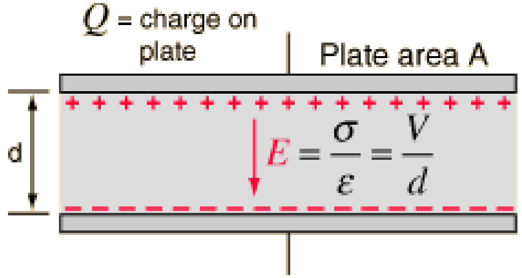


$$\boxed{P = \frac{1}{2} H^* E(Wd) = \frac{1}{2} E^* H(Wd)}$$

$$S \equiv \frac{1}{2} H^* E = \frac{1}{2} E^* H$$ (complex power density of electromagnetic field)

$$\boxed{\mathbf{S} = \frac{1}{2} \mathbf{E}^* \times \mathbf{H} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^*}$$ (Poynting vector = electromagnetic power density)

S: complex power density vector (in the direction of wave propagation)



- Stored energy in electric field

$$\boxed{w = \frac{1}{2}Cv^2 = \frac{1}{2}\varepsilon\frac{A}{d}(Ed)^2 = \frac{1}{2}\varepsilon E^2(Ad)} \text{: parallel-plate capacitor}$$

$$\boxed{w = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} = \frac{1}{2}DE} \text{ (J/m}^3\text{, electric field energy density)}$$

$$W = \frac{1}{4}DE^* = \frac{1}{4}E^*(\varepsilon' - j\varepsilon'')E = \frac{1}{4}(\varepsilon' - j\varepsilon'')|E|^2 \text{ (phasor for electric energy density)}$$

$$P = (j2\omega)W = \frac{1}{2}(\omega\varepsilon'')|E|^2 + j\frac{1}{2}(\omega\varepsilon')|E|^2 \text{ (phasor for electric power density)}$$

$$P_r = \frac{1}{2}(\omega\varepsilon'')|E|^2 = \frac{1}{2}\sigma|E|^2 = \frac{1}{2}JE \text{ (electric loss power density)}$$

$$P_i = \frac{1}{2}(\omega\varepsilon')|E|^2 \text{ (stored electric power density)}$$

- Stored energy in magnetic field

$$\boxed{w = \frac{1}{2}Li^2 = \frac{1}{2}\mu\frac{S}{d}(Hd)^2 = \frac{1}{2}\mu E^2(Sd)}: \text{ solenoid inductor}$$

$$\boxed{w = \frac{1}{2}\mathbf{B}\cdot\mathbf{H} = \frac{1}{2}BH} \text{ (J/m}^3\text{, magnectic field energy density)}$$

$$W = \frac{1}{4}BE^* = \frac{1}{4}H^*(\mu' - j\mu'')E = \frac{1}{4}(\mu' - j\mu'')|H|^2 \text{ (phasor for magnetic energy density)}$$

$$P = (j2\omega)W = \frac{1}{2}(\omega\mu'')|H|^2 + j\frac{1}{2}(\omega\mu')|H|^2 \text{ (phasor for magnetic power density)}$$

$$P_r = \frac{1}{2}(\omega\mu'')|H|^2 \text{ (magnetic loss power density)}$$

$$P_i = \frac{1}{2}(\omega\mu')|H|^2 \text{ (stored magnetic power density)}$$

3. Poynting's Theorem

- Power balance equation in electromagnetic field

$$P_s = P_r + P_l + j2\omega(W_m - W_e)$$

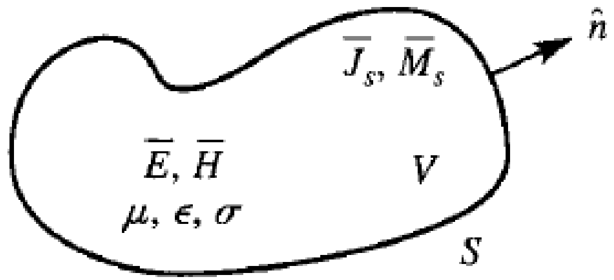
$$P_s = -\frac{1}{2}\int_V (\mathbf{E}\cdot\mathbf{J}_s^* + \mathbf{H}^*\cdot\mathbf{M}_s^*)dV : \text{ source power in } V$$

$$P_r = \frac{1}{2}\mathrm{Re}\oint_S \mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S} : \text{ power radiated through } S$$

$$P_l = \frac{1}{2}\int_V [(\sigma + \omega\varepsilon'')E^2 + \omega\mu'' H^2]dV : \text{power loss in } V$$

$$W_m = \frac{1}{4}\int_V \mu' H^2 dV : \text{ stored magnetic energy}$$

$$W_e = \frac{1}{4}\int_V \varepsilon' E^2 dV : \text{ stored electric energy}$$



4. Impedance and Admittance

- Impedance and admittance in terms of terminal current and terminal voltage

$$Z_{\text{in}} = \frac{2P_{\text{in}}}{|I|^2} = \frac{P_r + P_l + j2\omega(W_m - W_e)}{|I|^2/2} : \text{ input impedance}$$

I : terminal current

$$Y_{\text{in}} = \frac{2P_{\text{in}}}{|V|^2} = \frac{P_r + P_l + j2\omega(W_m - W_e)}{|V|^2/2} : \text{ input admittance}$$

V : terminal current

5. Wave Equation

- Electromagnetic field in charge-free region

$\rho = 0$ (in charge-free region)

$$\nabla \cdot \mathbf{E} = 0, \ \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \cdot \mathbf{H} = 0, \ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -j\omega\mu\nabla \times \mathbf{H} = \omega^2\mu\varepsilon\mathbf{E}$$

$\boxed{(\nabla^2 + \omega^2\mu\varepsilon)\mathbf{E} = 0}$: wave equation for electric field

Similarly,

$\boxed{(\nabla^2 + \omega^2\mu\varepsilon)\mathbf{H} = 0}$: wave equation for magnetic field

6. Planewave

- Planewave: one-dimensional wave

$$\frac{\partial}{\partial x}=0,\ \frac{\partial}{\partial y}=0,\ \frac{\partial}{\partial z}\neq 0$$

$$\nabla\cdot\mathbf{E}=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0\rightarrow E_z=0$$

$$\nabla^2=\frac{d^2}{dz^2}$$

$$(\nabla^2+\omega^2\mu\varepsilon)\mathbf{E}=\left(\frac{d^2}{dz^2}+\omega^2\mu\varepsilon\right)\mathbf{E}=0$$

$$\boxed{\mathbf{E}=\mathbf{E}^+e^{-jkz}+\mathbf{E}^-e^{+jkz}}$$

$\mathbf{E}^+e^{-jkz}$: wave propagating in $+z$ direction

$\mathbf{E}^-e^{+jkz}$: wave propagating in $-z$ direction

$\boxed{k=\omega\sqrt{\mu\varepsilon}=\frac{2\pi}{\lambda}}$ $(\mu,\varepsilon$ real) : propagation constant

λ: wavelength

- Intrinsic impedance

$$\mathbf{E} = E_x\,\hat{\mathbf{x}}\,e^{-jkz}$$

$\hat{\mathbf{k}} = \hat{\mathbf{z}}$: direction of propagation

$$\mathbf{H} = -\frac{1}{j\omega\mu}\nabla\times\mathbf{E} = -\frac{1}{j\omega\mu}\left(\hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\times\mathbf{E}$$

$$= -\frac{1}{j\omega\mu}\hat{\mathbf{y}}(-jk)E_x\,e^{-jkz}$$

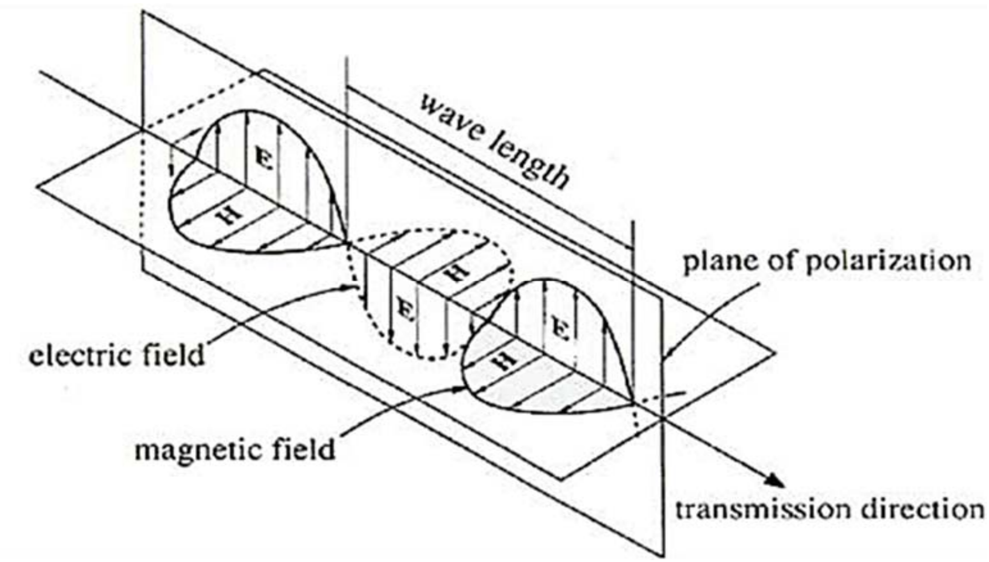
$$\mathbf{H} = \frac{k}{\omega\mu}E_x\,\hat{\mathbf{y}}\,e^{-jkz} = \frac{1}{\eta}E_x\hat{\mathbf{y}}e^{-jkz}$$

$$\boxed{\mathbf{H} = \frac{\hat{\mathbf{k}}\times\mathbf{E}}{\eta}}$$

$$\boxed{\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}}$$: intrinsic impedance

$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.73\ \Omega}$$ (in vacuum)

$$\boxed{\mathbf{E} = \eta\mathbf{H}\times\hat{\mathbf{k}}}$$



7. Coding Example

Planewave

Input: f, u_r, ε_r

Calculate: k, λ, η

$$\omega = 2\pi f$$

$$\mu = \mu_0 \mu_r,\ \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

$$\varepsilon = \varepsilon_0 \varepsilon_r,\ \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$$

$$k = \omega\sqrt{\mu\varepsilon}$$

$$\lambda = 2\pi / k$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Fin

(End)